STATIC EQUILIBRIUM

About this unit

This is a foundation unit, and a lot of the work should be familiar to you already. If you are confident about working with forces and moments you may like to move forwards to the sections on non-concurrent forces, and the turning moment of a crank mechanism.

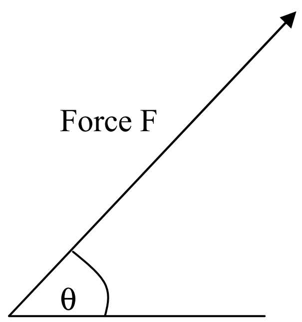
FORCE

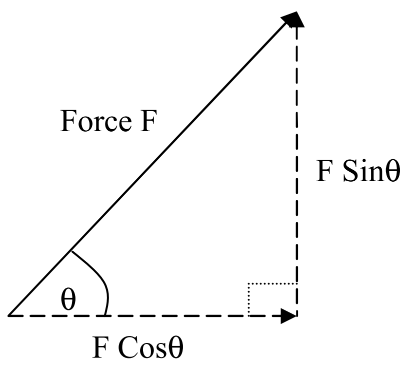
The presence of a force can be detected from its effects. A force can be exerted to lift something, start an object moving, stop the movement of an object or body, or cause an acceleration. Additionally a force can be compressive, tensile, shearing etc.

The base unit of force is the "Newton", defined as the force that will give an acceleration of one m/s^2 to a mass of one kilogram. The Newton is a derived unit of ($kg.m/s^2$).

Force is a vector quantity so it defined by both its line of action and its magnitude. An example is the gravitational force is exerted on the mass of a body, given by Mg, this acts towards the centre of the earth, but is usually considered to be acting downwards.

A body mass of 25 kg would have an mg value of 25 *x* 9.91 = 245.25 (N), this would be known as the weight of the body.





If "F" had a value of 20 kN acting north east, its rectangular components (or its vertical and horizontal components) are:-

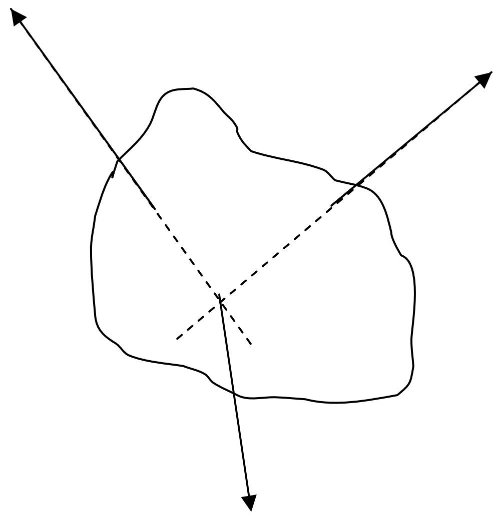
20 Sin θ acting due north = 14.142 kn.
20 Cos θ acting due east = 14.142 kn.

The components of a force act at the same point as the applied force. **A force can have no component at right angles to its line of action,** e.g. a horizontal force has no vertical component.

Resultant of more than two forces

If more than two forces act at the same time at the same point and they act in directions which lie in the same plane, they are known as CO-PLANAR forces.
If all the **lines of action** of the forces pass through the same point, they are known as a system of CONCURRENT forces.

Concurrent forces



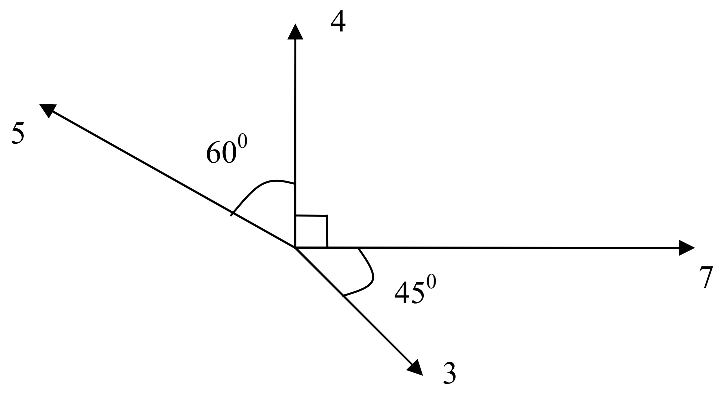
The resultant of such a system of forces can be found by vector addition, i.e. by drawing the forces 'nose to tail' and forming a "Polygon of Forces". A polygon is a closed figure bounded by a number of straight lines. It is a general mathematical term which covers all such cases.
For example, a square is a 4 sided polygon, a hexagon is a 6-sided polygon.

An alternative to drawing the force polygon to scale is to resolve each force into its' horizontal and vertical components. We can then simply summate all the horizontal forces together and all the vertical forces together. The resultant force is then found by combining these two forces together, forming a right-angle triangle with the resultant as the hypotenuse.

Example 1.

Find the resultant of the concurrent co-planar system of forces shown. All Forces are given in Newtons.

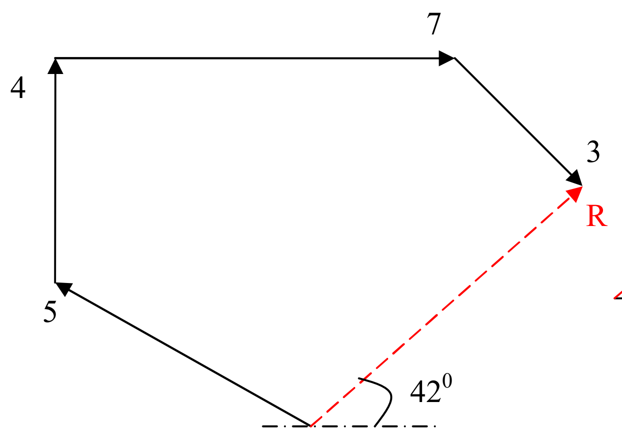
Space Diagram



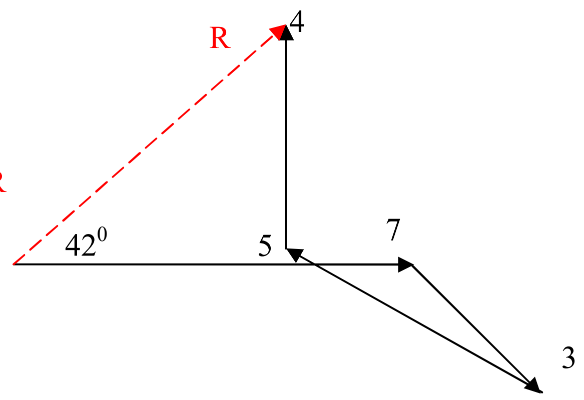
Solution.

Draw the vectors nose-to-tail, and the resultant force is the force that goes from the **start to the finish** of the diagram. Note that it does not matter which order we take the forces in. Although the diagram will look different, the resultant force will be the same in each case.

Vector Diagram



Alternative Vector Diagram



The resultant force is 6.5N at 42^o to the horizontal as shown. Note that although the vector diagrams can be drawn with the forces in any order (two examples shown) the resultant will be the same in magnitude and direction. Draw these to scale to prove the result to yourself. Even if you are going to calculate an answer, it pays to draw the diagram approximately to scale to verify your answer. Remember, you don't always have time in an exam to do the calculation, and a good drawing will get you a good proportion of the marks.

EQUILIBRIUM

A body is said to be in static EQUILIBRIUM under a system of forces if there is no resulting tendency for the body to move.

For a body in equilibrium the resultant of any forces acting on it must be zero. Additionally if the forces are not concurrent, [if their lines of action do not all pass through the same point] then all the moments of the forces must be balanced.

The definition of static equilibrium is:-
"The sum of the forces along any line must equal zero and the sum of the moments about any point must equal zero".

For the concurrent co-planar force systems being considered, if the body on which they act is in equilibrium their resultant must be zero. i.e. the polygon of forces for the system must close, with the 'heads' and 'tails' of the vectors joining all the way round.

Equilibrant:

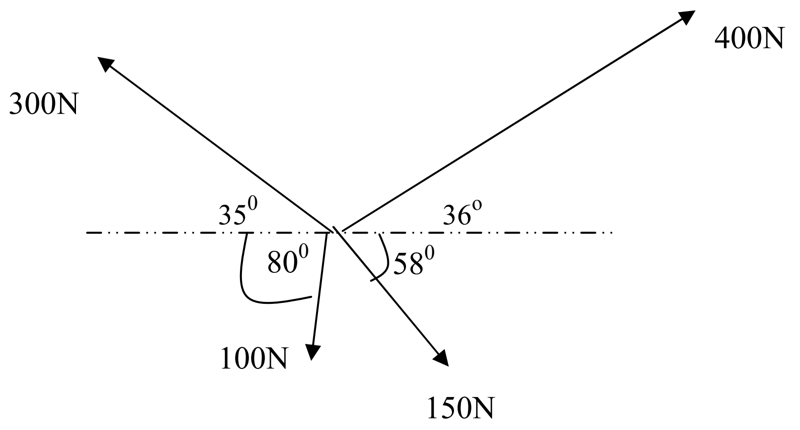
If a system of concurrent, co-planar forces is in equilibrium, their resultant must be zero. If a system of such forces has a resultant, then an additional force, the EQUILIBRANT for the system, is required to maintain equilibrium.

This equilibrant force must be numerically equal to the resultant, and must act in the opposite direction, along the same line of action. Note the equilibrant must just balance the resultant. If the resultant is a vector R, the equilibrant is the vector - R.

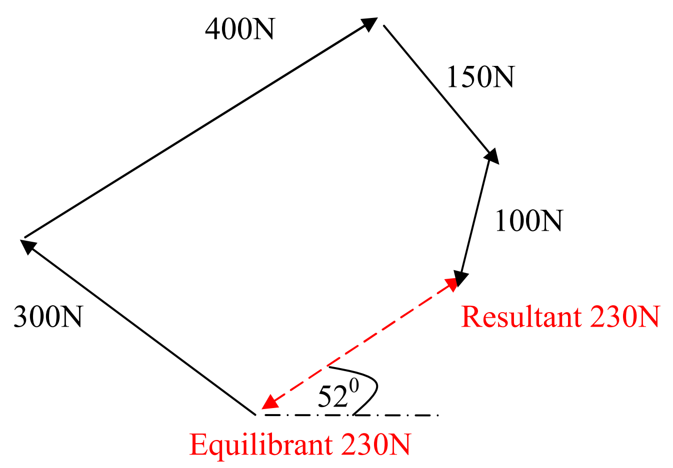
Example 2

A system of concurrent, co-planar forces is shown, determine the value of the single force needed to bring the system into equilibrium.

Space Diagram



Vector Diagram



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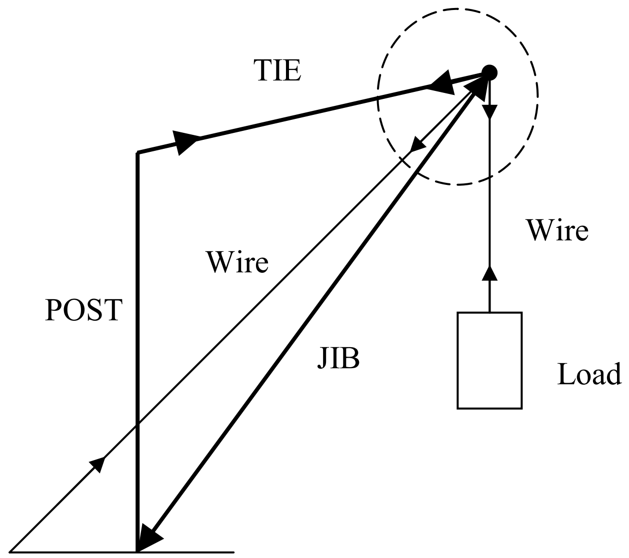
So the single force needed to bring the system into equilibrium is 230N, at an angle of 128^0 clockwise from the horizontal, as shown.

JIB CRANES

We shall now apply force vector theory to some specific force systems, starting with Jib cranes. The simplest jib-crane consists of a post, a jib and a tie. The post is usually vertical, with the jib hinged at its lower end to the bottom of the post. The tie connects the top of the jib to the top of the post.

The junction of the tie and jib is called the head. In problems on jib cranes it is often taken that the load is suspended directly from a fixture at the crane head, the problem then involves a simple triangle of forces.

In other cases the description may include a pulley at the crane head, the lifting rope would pass over this pulley and down to a winch behind the crane. Such cases involve more than three forces at the crane head, and such a case is illustrated below.



SPACE DIAGRAM

The purpose of the space diagram is to find the angles so that we can draw the vector diagram. We should also at this point decide which members are struts (in compression, arrows go OUTWARDS) and which are ties (in tension, arrows go INWARDS). We should circle the point under consideration for drawing the vector diagram. This helps to remind us that the arrows on the vector diagram should be in the same direction as the arrows on the space diagram **at the point under consideration**.

Example 3.

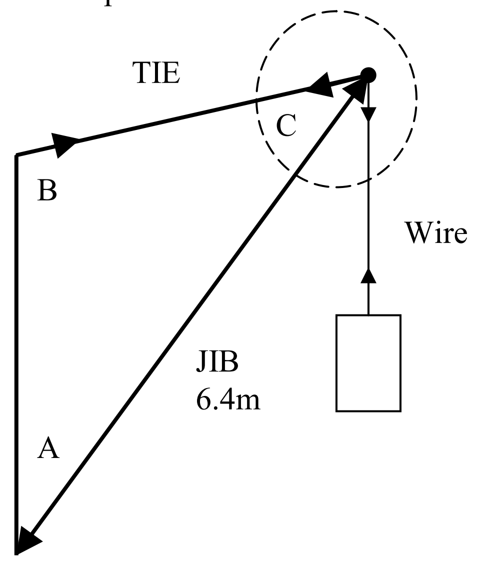
The jib length of a simple jib crane is 6.4 m, the vertical post is 4 m and the tie length is 3.5 m.

Find the forces in the jib and tie when a load of 40 kN is suspended from the crane head.

Solution

At the crane head, three forces meet which are in equilibrium. The jib is under compression and therefore a strut. The tie is in tension and as its name suggests is a TIE. The wire is under tension (Wires are always under tension, you can't push on a rope can you!). The wire is fixed at the crane head and therefore there are only three forces acting at the crane head

The arrows are inserted according to this philosophy and the vector diagram is constructed to represent the forces at the crane head.



SPACE DIAGRAM

$$\text{Cos A} = \frac{b^2 + c^2 - a^2}{2 \times b \times c} = \frac{6.4^2 + 4^2 - 3.5^2}{2 \times 6.4 \times 4}$$

$$= 0.8732 \quad \text{hence:-} \quad \underline{A = 29.167^o}$$

Sin Rule,

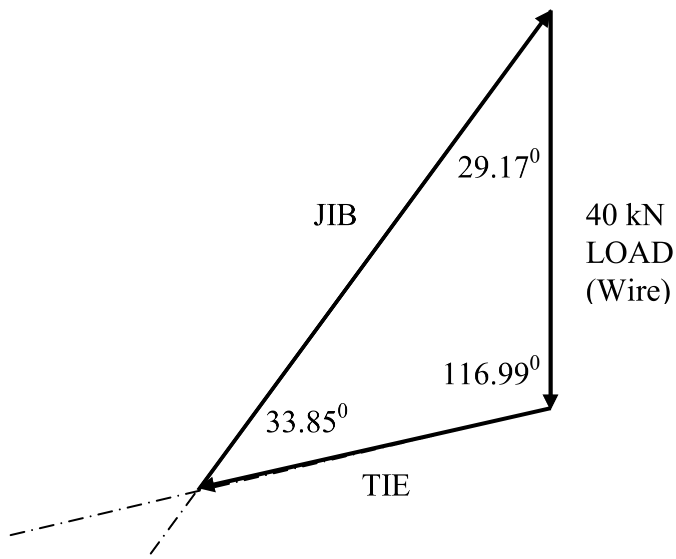
$$\frac{3.5}{\text{Sin } 29.167} = \frac{4}{\text{Sin C}}$$

$$C = \text{Sin}^{-1} \frac{4 \times \text{Sin } 29.167}{} = \text{Sin}^{-1}\, 0.557 = 33.85^o$$

3.5

$$B = 180 - (29.167 + 33.847) \quad = 117^{o}$$

Having drawn the space diagram and calculated the angles we can now draw the vector diagram. Remember we can draw the vectors in any order, but we should start with those we know most about, i.e. magnitude and direction. In this case this is the load. We can then draw either the jib force or tie force, in direction only, ensuring the arrows follow round "nose-to-tail". The line will be of indeterminate length. For equilibrium, we know that the diagram must form a closed figure, so we can draw the remaining vector backwards form the start point, again using its direction only. The intersection of this line with our other line enables us to determine the magnitude of each force as shown. Once you have drawn the vector diagram, check the arrows. They should all follow round for a system in equilibrium, and they should be in the same direction as those on your space diagram **at the point under consideration**. If this is not the case, then you have made a mistake, or possibly what you considered to be a strut in your space diagram might be a tie, or vice-versa.



VECTOR DIAGRAM

Sin Rule,

$$\frac{\text{Jib Force}}{\quad} = \frac{40}{\quad}$$

Sin 116.985 Sin 33.847

$$\text{Jib Force} = \frac{40 \, x \text{ Sin } 116.985}{\text{Sin } 33.847} = \underline{63.997 \text{ kN}}$$

$$\frac{\text{Tie Force}}{\text{Sin } 29.167} = \frac{40}{\text{Sin } 33.847}$$

$$\text{Tie Force} = \frac{40 \, x \text{ Sin } 29.847}{\text{Sin } 33.847} = \underline{35\text{kN}}$$

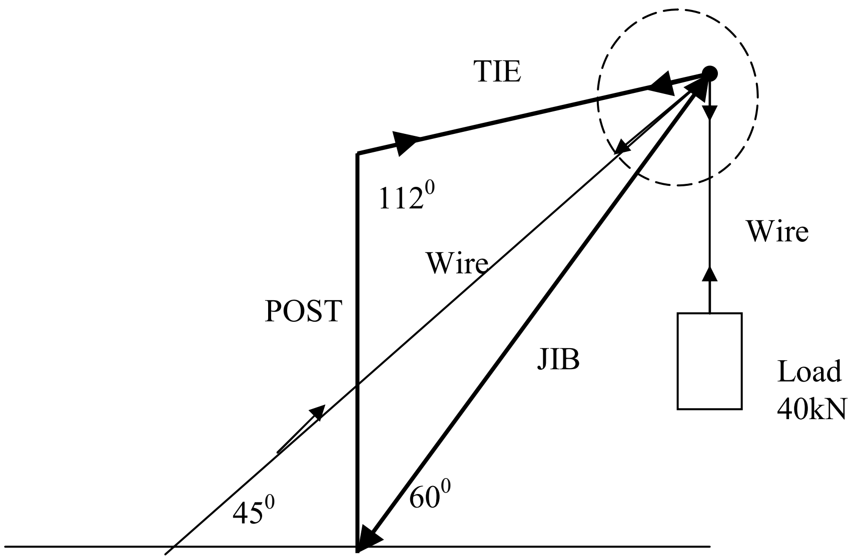
Example 4

The jib of a crane is 6.5m long and is hinged at its foot to the base of a vertical post 4m high, the jib being at an angle of 30^0 to the post. A lifting wire passes over a pulley at the crane head and is led at 45^0 to a winch. The wire is in the plane of the jib. Determine the forces in the jib and tie when a 40kN load is suspended.

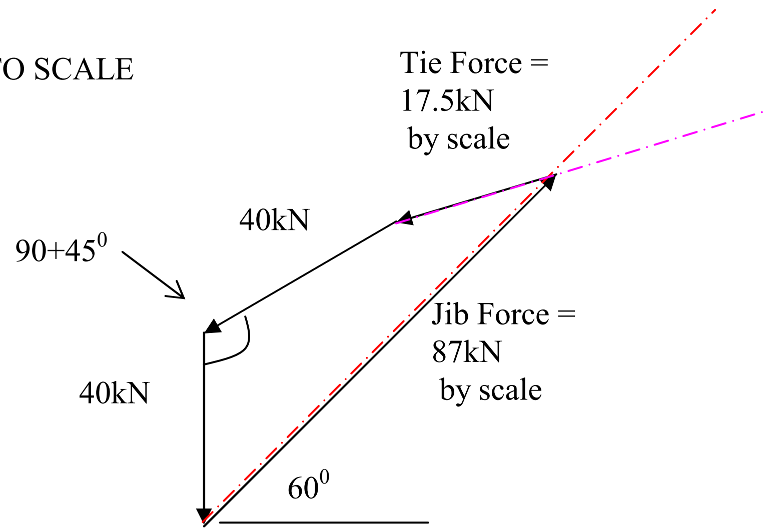
Solution

This problem involves four forces at the crane head, the additional force being the force in the wire passing over the head and back to the winch. Note that if there is no mention of friction in the question, then we can assume that the force in the wire is the same from one side of the pulley to the other, and this means the force in the wire on each side of the pulley will be the load, 40kN. Since we know the magnitude and direction of these forces, we should draw these first. Note that the question said "Determine" and this clearly allows us to use the much quicker method of using a scale drawing of both the space and vector diagrams.

SPACE DIAGRAM, DRAWN TO SCALE



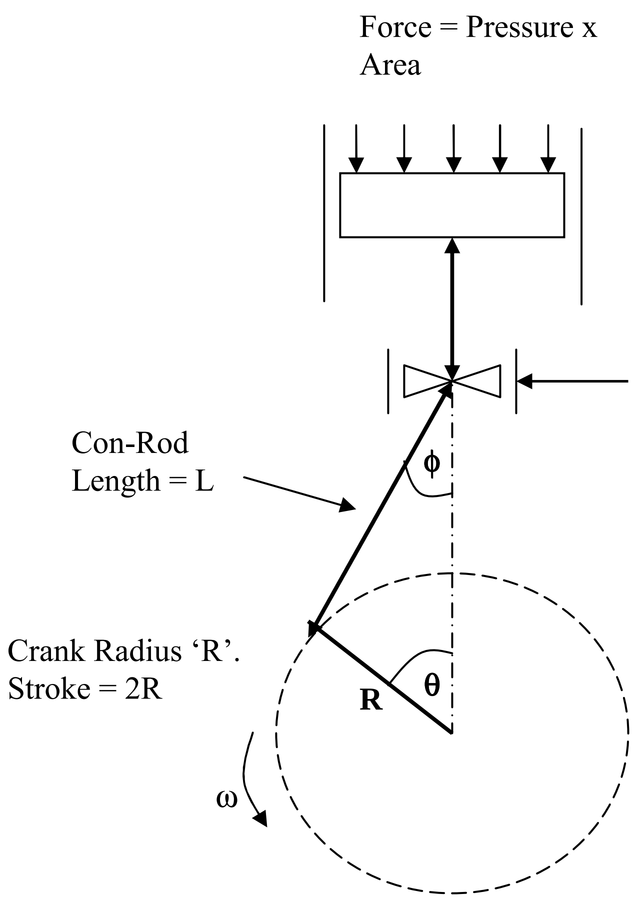
VECTOR DIAGRAM, DRAWN TO SCALE



RECIPROCATING ENGINE MECHANISM

A Force system which is in everyday use is the reciprocating engine mechanism. The connecting rod and crank of a reciprocating engine converts the reciprocating motion of the piston to rotary motion at the crank shaft.

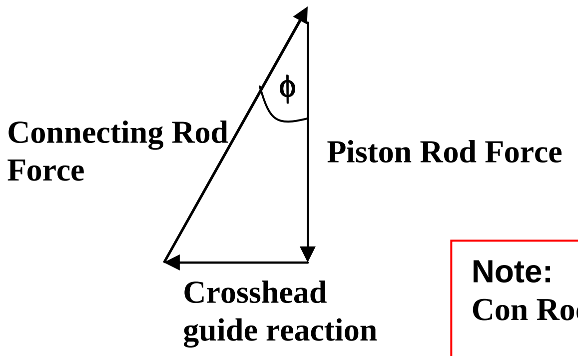
Consider the forces meeting at the crosshead, the lower end of the piston rod pushes vertically downwards on the crosshead, the thrust in the connecting rod is an upward resisting force at its top end inclined to the vertical, the guide exerts a horizontal force to balance the horizontal component of the con rod thrust.



Note that for any crank angle θ, from Sine Rule,

$$\frac{R}{\sin\phi} = \frac{L}{\sin\theta}$$

Note that the if the piston effort acts vertically, the guide force acts horizontally, hence the vector diagram of the forces at the crosshead is always a right-angled triangle.



Note:
Con Rod Force = $\dfrac{\text{Piston Rod Force}}{\text{Cos } \phi}$

Example 5

The piston of a reciprocating engine exerts a force of 160 kN on the crosshead when the crank is 45 degrees past top dead centre. If the stroke of the piston is 950 mm and the length of the connecting rod is 1.6 m, find the guide force and the force in the connecting rod.

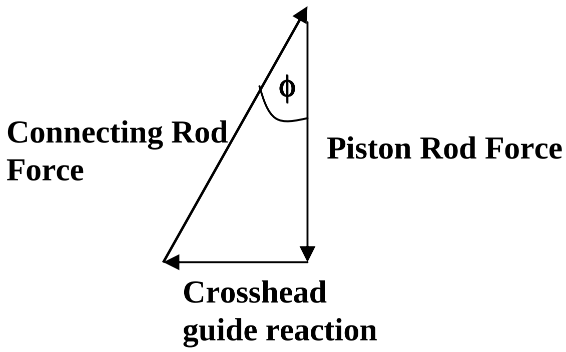
crank length = ½ stroke = 0.475 m

length of connecting rod = 1.6 m

crank angle from T.D.C. = 45^o

$$\frac{0.475}{\text{Sin } \phi} = \frac{1.6}{\text{Sin } 45}$$

$\phi = 12.12^o$



From vector diagram:-

$$\text{Tan } \phi = \frac{\text{Guide Force}}{\text{Piston Force}}$$

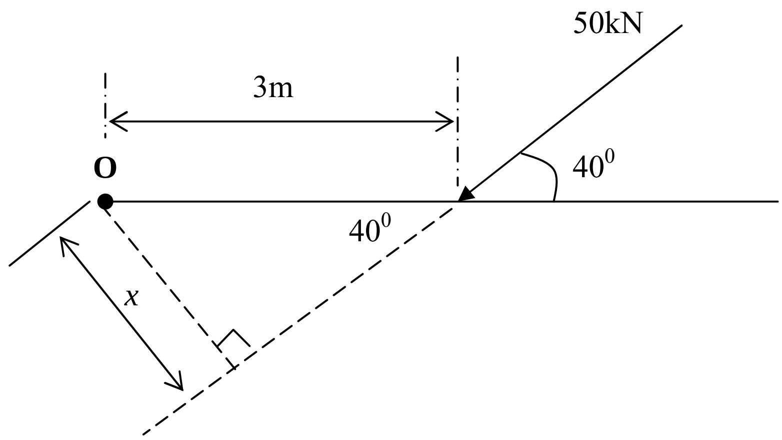
Guide force = 160 x Tan ϕ = <u>34.353 kN</u>

$$\text{Cos } \phi = \frac{\text{Piston force}}{\text{Con Rod Force}}$$

$$\text{Con Rod Force} = \frac{160}{\text{Cos } 12.12} \quad = \underline{159 \text{ kN}}$$

MOMENT OF A FORCE

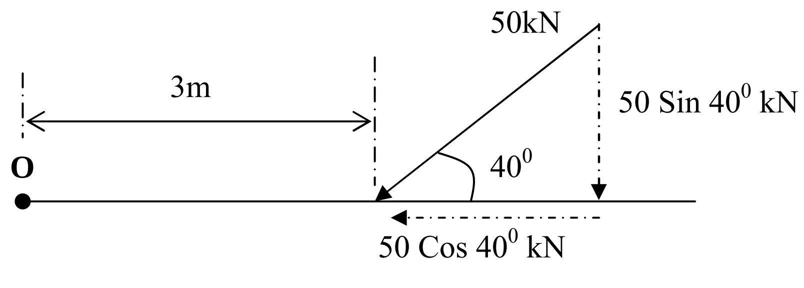
You should recall that the moment of a force "P" about a point "O", is given by the product of the force and the **perpendicular** distance of its line of action from "O".



For instance, for the arrangement shown, the moment of the 50kN force about 'O' is

Moment $= 50x = 50 \text{ x } 3\text{Sin } 40 = 32.14$ kNm

Note that the same result will be arrived at if we take the component of the 50kN Force which is perpendicular to the point under consideration, as below.



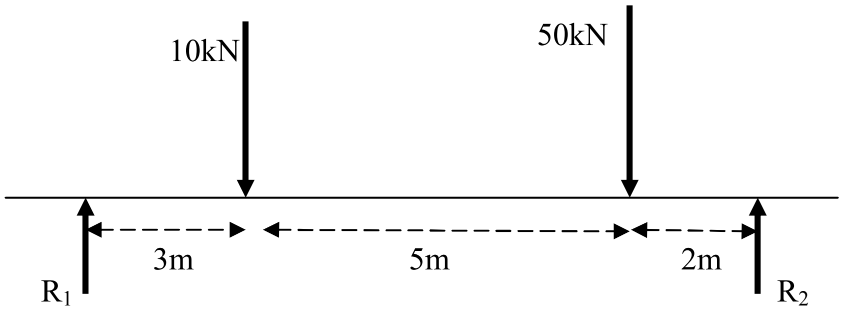
Note that the components of the 50kN force are still **applied at the same point** as the 50kN Force itself.

So for the vertical component, moment $= 50 \text{ Sin } 40^0 \text{ x } 3 = 32.14$ kNm as before

Note that the line of action of the horizontal component, $50 \text{ Cos } 40^0$ passes through the point we are taking moments about, and therefore has no moment about this point.

Example 6

Calculate the reactions at each of the supports for the simply supported beam shown.



Solution

One of the benefits of taking moments is that it enables us to eliminate the moment of at least one of the forces by taking moments about where that force acts. So here, we would wish to eliminate either one of the unknown reaction forces, so let us take moments about R1, remembering that for equilibrium, the sum of the forces about any point must equal zero.

Summing the moments about R_1, clockwise positive, summation = zero. (this statement is often written shorthand, as follows).

Σ M clockwise +ve, $\Sigma = 0$

$10 \times 3 + 50 \times 8 - R_2 \times 10 = \text{Zero}$

$R_2 = 43$ kN

We could now take moments about R_2 and find R_1, but it is quicker to apply the other condition for equilibrium, that the sum of the forces in any direction must be zero.

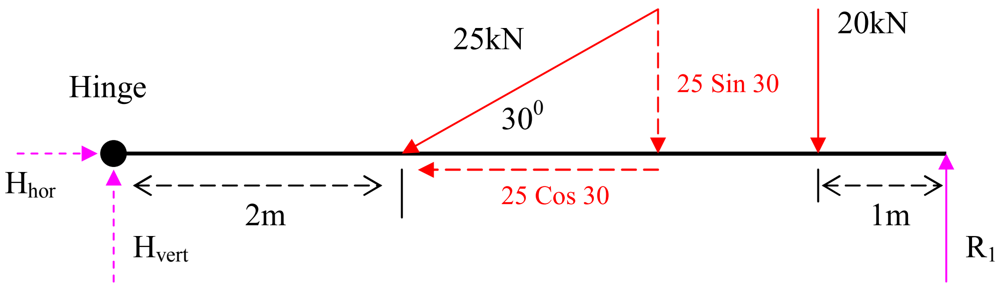
Σ Vert Forces = zero:

$R_1 + R_2 - 60 = \text{zero}$

$R_1 = 60 - 43 = 17$ kN

Example 7

A beam, 8 m long is hinged at one end and simply supported at the other. It is loaded as shown.
Determine the magnitude and direction of the reaction force at the hinge.



Solution

An important principle illustrated in this example is that reactions at simply supported flat surfaces are always normal to the surface (as with R), but we do not know exactly what direction the reaction at a hinge or pin joint will be and therefore we need to calculate its components, usually horizontal and vertical, separately and then add them up to give the magnitude and direction of the total reaction at the hinge.

In many mechanics questions we can rely on the basic conditions of equilibrium to produce three equations from which we can solve for at least 3 unknowns. We can summate the forces in two, mutually perpendicular directions and summate the moments. In addition, by taking moments about an unknown force we can eliminate its moment at that point and hence "ignore" it from the point of view of its moment (But remember to consider it as a force when summating forces! – many students eliminate the moments at hinges by taking moments about them but then forget that this does NOT mean there is no force at the hinge).

So, let us apply the three conditions for equilibrium here.

"Summation of the Horizontal Forces, left to right positive = zero".

Which can be written in shorthand form:

Σ H Forces → +ve, $\Sigma = 0$

$H_{HOR} - 25 \text{ Cos } 30 = 0$

So $H_{HOR} = 25 \text{ Cos } 30 = 21.651$ kN

Summation of the Horizontal Forces, left to right positive = zero

Σ V Forces ↑ +ve, $\Sigma = 0$

$H_{VERT} + R_1 - 25 \text{ Sin } 30 - 20 = 0$

$H_{VERT} + R_1 = 32.5$ kN ……………………..1

This of course leaves us with two unknowns, so we still need a second equation. So let us take moments, and since hinge reactions are always awkward because we do not, at this time, know their line of action, let us take moments about the hinge and thus make the moment of the hinge force about this point zero.

"Sum of moments about the hinge, clockwise positive, summation = zero"
Which can be written in shorthand form:
Σ M about hinge, clockwise +ve, $\Sigma = 0$

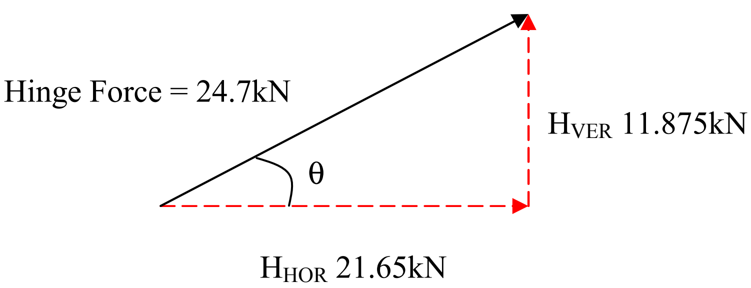
$$25 \text{ Sin}30 \times 2 + 20 \times 7 - R_1 \times 8 = 0$$

$$R_1 = \frac{25 + 140}{8} = 20.625\text{kN}$$

From equation 1, $H_{VERT} + R_1 = 32.5$ kN.

Substituting for R1 gives $H_{VERT} = 11.875$ kN

We can now combine the horizontal and vertical components of the hinge force by adding them vector ally (drawn nose-to-tail remember) to find the magnitude and direction of the hinge force.



$$\text{Tan } \theta = \frac{11.875}{21.65\text{kN}}$$

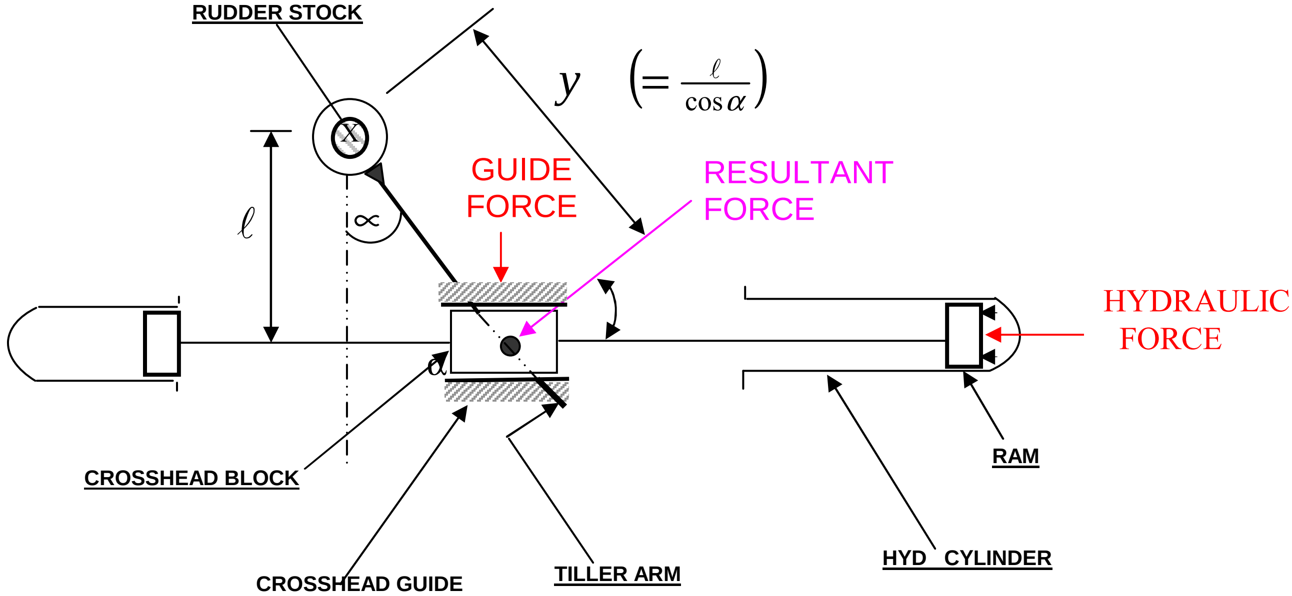
$$\theta = 28.74^0$$

From Pythagoras,

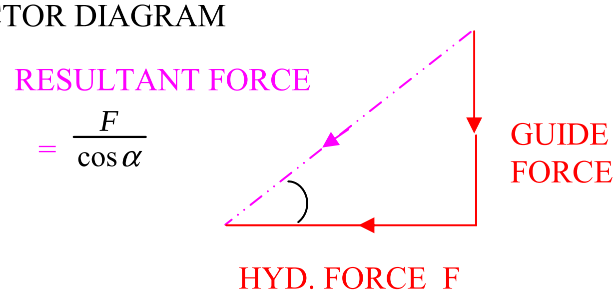
$$HingeForce = \sqrt{11.875^2 + 21.655}^2 = \underline{24.69 \text{ kN at } 28.74^o} \text{ as shown}$$

THE RAPSON'S SLIDE (STEERING GEAR)

An important mechanism which uses the principles of moments we have just discussed is the Rapson's slide, used in ram type steering gears. The hydraulic pressure in the rams generates a force (equal to pressure x area) on the crosshead block. This can be resolved into two forces, one normal to the guide, and the other perpendicular to the tiller arm. It is this latter force that will give a turning moment to the rudder.



VECTOR DIAGRAM



Turning moment on rudder stock = Resultant force × y

Turning moment on rudder stock $= \frac{F}{\cos\alpha} \times \frac{\ell}{\cos\alpha} = \frac{Fl}{\cos^2\alpha}$

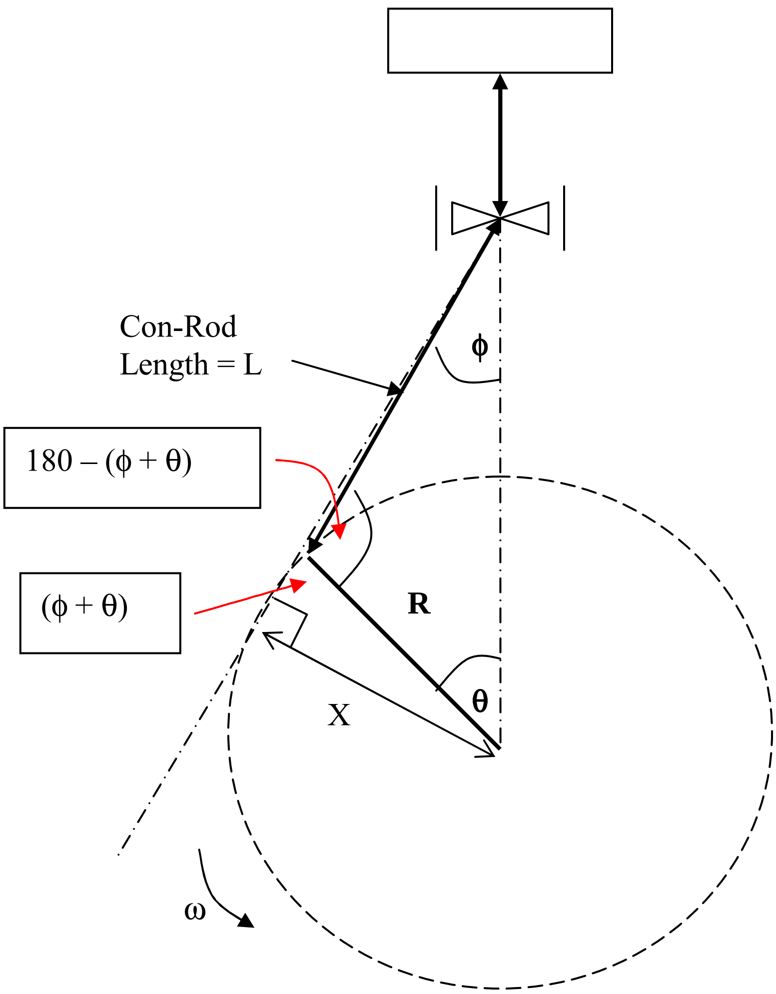
This is for a two ram gear, for a 4 ram gear we have double this, i.e. $\frac{2Fl}{\cos^2\alpha}$

TURNING MOMENT OF A CRANK MECHANISM

Now that we have recapped moments, let us take another look at the crank mechanism, this time in terms of the turning moment produced at the crankshaft.

From previous work, $\text{Con-Rod Force} = \dfrac{\text{Piston Rod Force}}{\cos\phi}$

The turning moment is therefore given by the product of this con-rod force and its lever arm 'X'.



From the diagram, $\sin(\phi + \theta) = X/R$. So turning lever 'X' $= R\sin(\phi + \theta)$

Turning effect $=$ Con-rod force x lever arm $= \dfrac{\text{Piston Rod Force} \times R\sin(\phi+\theta)}{\cos\phi}$

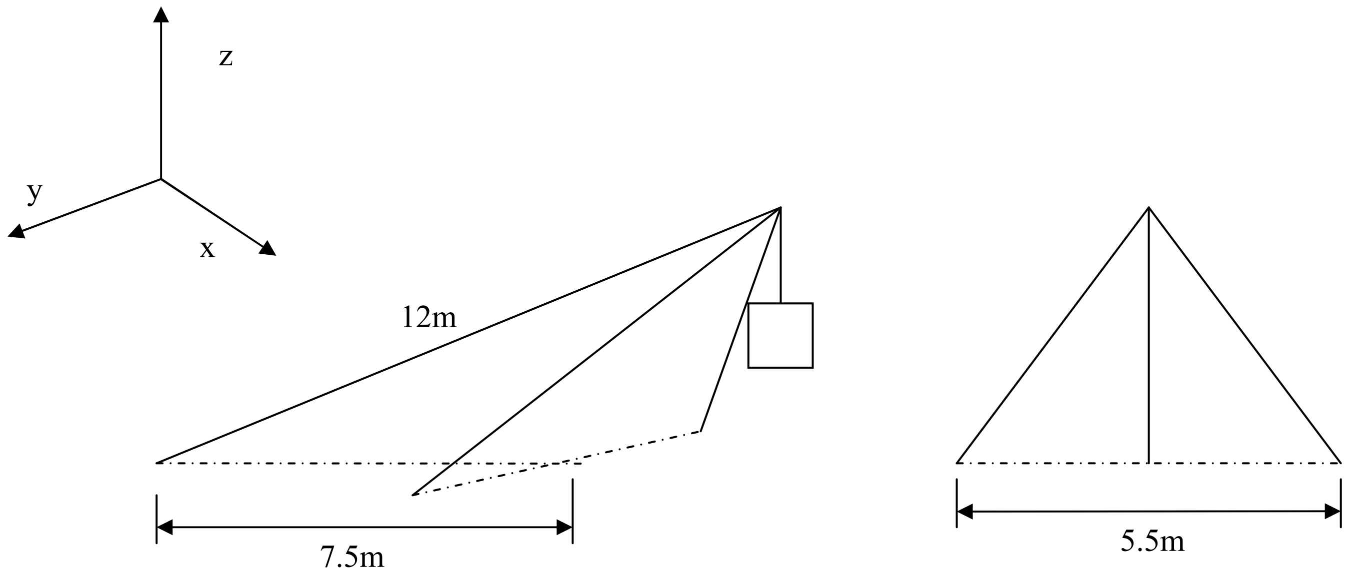
***This is an important result, which you should be able to prove*.**

Non-Coplanar Forces

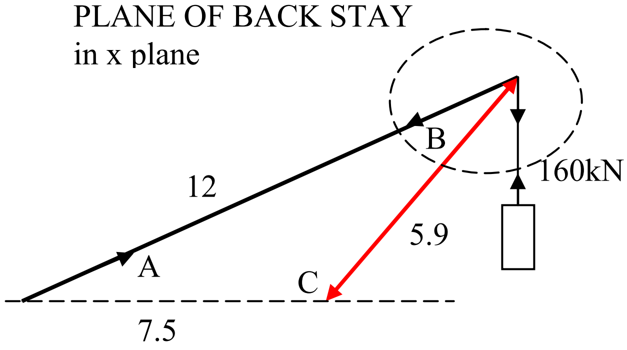
If the forces are not in the same plane, then we cannot draw a two dimensional vector diagram to represent them. We must therefore apply a different technique. For the common arrangement of shear legs, the method is to use an imaginary component, which has the advantage of being in the same plane as all of the other components in the system. This is best illustrated by an example.

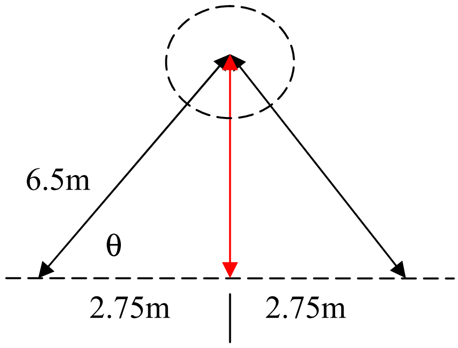
Example 8.

A set of shear legs arranged as shown are 6.5 m long, the back stay is 12 m long and the distance from the rear of the backstay to the centre of the shear leg straddle is 7.5 m.
Calculate the force in each shear leg when a load of 160 kN is supported.



Solution
The first step is to replace the two front legs with one, central **imaginary** leg I_L. This will be **in the same plane as both the back legs and the front legs**. We can now draw a space diagram for each plane.





Imaginary leg length I_L

$= \sqrt{6.5^2 - 2.75^2}$

$= 5.9m$

The purpose of the space diagram is to find the angles so that we can draw the vector diagram. We should also at this point decide which members are struts (in compression) and which are ties (in tension). We can also circle the point under consideration for drawing the vector diagram. This helps to remind us that the arrows on the vector diagram should be in the same direction as the arrows on the space diagram at the point under consideration.

From the space diagram, the angles can be found as follows:

From the Cosine rule,

$$\text{Cos A} = \frac{12^2 + 7.5^2 - 5.9^2}{2 \times 12 \times 7.5}$$ Which gives A = 23^0

From the Sin Rule,

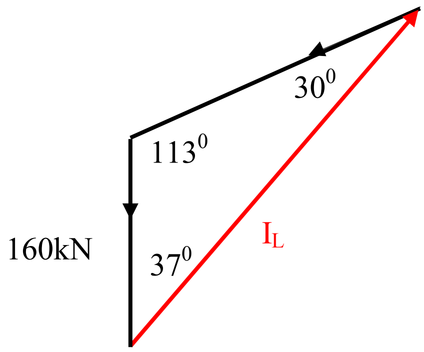
$$\frac{5.9}{\text{Sin } 23.1} = \frac{7.5}{\text{Sin B}}$$ Which gives B = 30^0

This makes the remaining angle C = 127^0

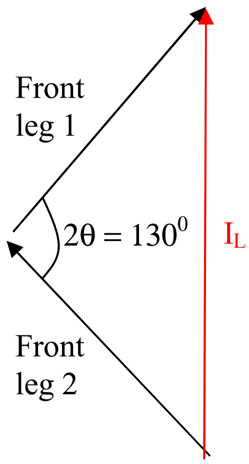
$$\text{Cos } \theta = \frac{2.75}{6.5}$$ Which gives θ = 65^0

We can now draw the Vector diagrams, firstly for the plane of the backstay. **Remember!** The direction of the vectors is given by the direction of the force in the member at the point under consideration, and for equilibrium the vectors are drawn nose-to-tail and form a closed figure.

VECTOR DIAGRAM,
PLANE OF BACK STAY
in x plane



VECTOR DIAGRAM,
PLANE OF FRONT
LEGS in y plane



From the vector diagram for the plane of the backstay we can calculate the load in the imaginary leg using the Sin rule:

$$\frac{I_L}{\sin 113^0} = \frac{160}{\sin 30^0}$$

Which gives I_L Force = 295 kN

We can now transfer this force into the plane of the FRONT legs. Remember that in this plane the vector will be a vertical line. The real front legs combine together to give the same **RESULTANT** as the imaginary leg. So note that in this second vector diagram we are not drawing three forces meeting at a point, we are drawing the two real front leg forces added together to give a resultant equal to the imaginary leg force.

From this second vector diagram, again using the Sin Rule,

$$\frac{295}{\sin 130^0} = \frac{\text{Front Leg Force}}{\sin 25^0}$$

Which gives Force in each front leg = <u>163 kN</u>

Self Assessed Questions for you to try. (Answers given)

1 The following four forces pull on a point: 70 N due East, 100 N, East 35^o North, 45 N, West 15^o North and 80 N, South West. Find the resultant force (a) graphically (b) by calculation, and state its direction relative to the 70 N force.

Ans: 53 N, East 13.5^o North

2 A derrick is hinged at its lower end to the bottom of a vertical mast and supported at its upper end by a horizontal topping lift attached to the mast. If the angle between derrick and mast is 40^o calculate the forces in the derrick and topping lift when a load of 60 kN is suspended from the derrick head.

Ans: 78.3 kN Compressive 50.2 kN Tensile

3 A simple jib crane has a vertical post 2 m high, the jib is 4 m long and the tie is 3 m long. A wire rope is attached to the jib head and is led downwards and outwards from the crane to make an angle of 60^o to the vertical. Find, by graphical construction, the forces in the jib and tie when the pull in the rope is 75 kN.

Ans: Jib 108.5 N Tie 148.5 N

4 A boiler with a mass of 20 tonne is lifted by a pair of sheer-legs whose lengths are 20 m. These legs are inclined forwards at 60^o to the horizontal and are 7 m apart at the base, being supported at the rear by a back leg of length of 30 m. Find the force in each leg.

Ans: Backstay 229.5 kN Legs 191.5 kN

5 A uniform beam is lifted horizontally by a sling. In one case the sling is 1.5 m from one end where a force of 320 N is required to keep the beam horizontal. In another case, the sling is 1.2 m from the same end when a force of 500 N is required to keep the beam horizontal. Find the weight and length of the beam.

Ans: 400 N 5.4 m

6 A uniform beam is 10 m long and has a mass of 2.55×10^3 tonne. If a vertical force of 100 kN is applied 2.5 m from the support at one end, how far from the support at the other end should a vertical force of 150 kN be applied so that the force in each support is the same? Ans: 3.33 m

7 A connecting rod with a mass of 0.714 tonne is 3 m long and its centre of gravity is at mid-length. If it hangs suspended from the top end, find the horizontal force required at the bottom end to hold the rod at 20° to the centre-line of the engine. Find also the minimum force required and the magnitude and direction of the reaction at the crosshead when the minimum force is applied.

Ans: 1.274 kN 1.2 kN 6.69 kN at 9.7° to vertical

8 A beam of length 10 m with one end resting against a smooth inclined plane of 45° is held horizontally between a pair of frictionless rollers at the other end. A horizontal force of 4 kN is applied at the end held between the rollers whilst a vertical force of 8 kN acts downward at 2 m from the end resting against the incline. Determine, for equilibrium conditions, the magnitude of a force F acting at 2 m from the rollers and at 60° downward from the horizontal, away from the rollers.

Ans: 7.36 kN