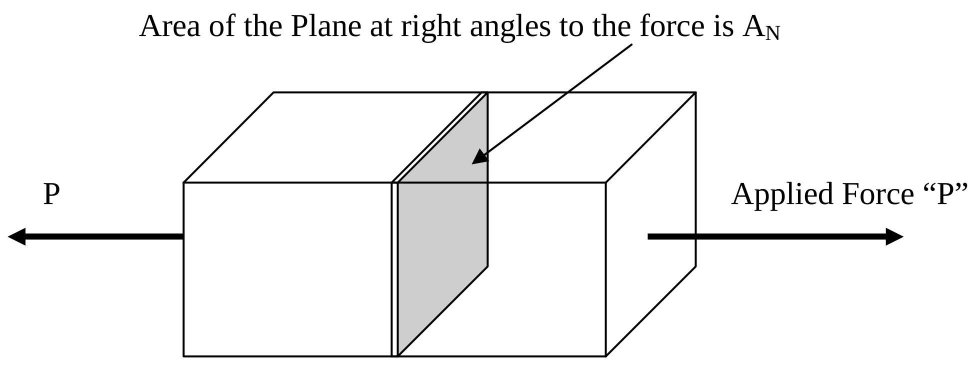
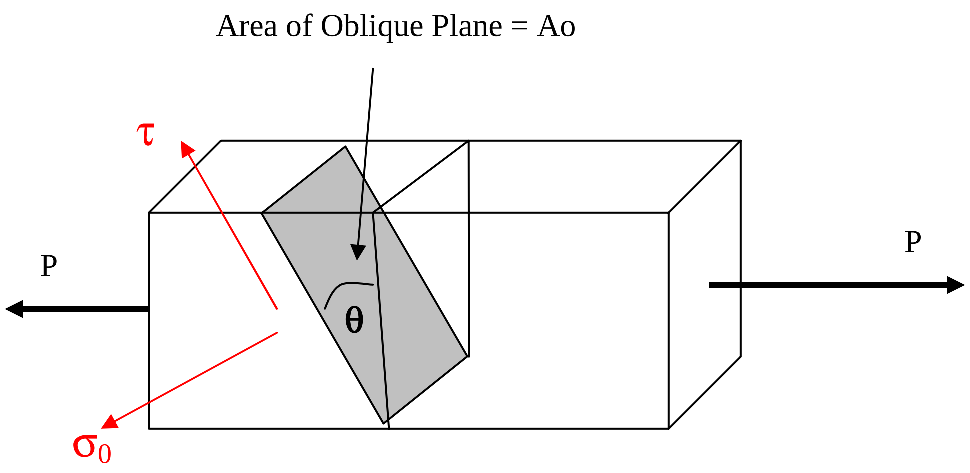
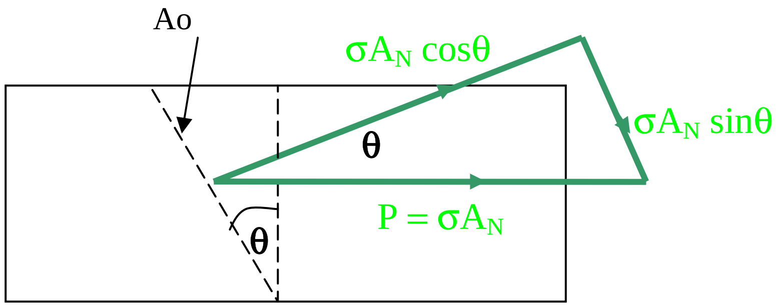
STRESSES ON OBLIQUE PLANES

In our work so far on mechanics we have usually only considered the plane at right angles to the load as shown.



If we now consider a plane at some other angle θ to the normal, then the load can be resolved into two mutually perpendicular directions, parallel and normal to the plane. The component normal to the oblique plane will cause a direct stress $\boldsymbol{\sigma_o}$, whilst that parallel to the plane will cause a shear stress $\boldsymbol{\tau}$.





Note that the area of the oblique plane Ao is greater than the area of the normal plane A_N, since

$$Ao = \frac{A_N}{\cos\theta}$$

So since stress is load/area, the two stresses on the oblique plane are:-

1. Normal to the oblique plane, a tensile stress,

$$\sigma o = \frac{\sigma A_N \cos\theta}{A_N/\cos\theta} = \sigma \cos^2\theta$$

2. Parallel to the plane, a shear stress, t, where

$$\tau = \frac{\sigma A_N \sin\theta}{A_N/\cos\theta} = \sigma \sin\theta \cos\theta$$

You may see this latter equation written in a different form. Remember from trigonometric identities,

$$\sin\theta \cos\theta = ½ \sin 2\theta, \text{ so}$$

$$\tau = ½\, \sigma \sin 2\theta$$

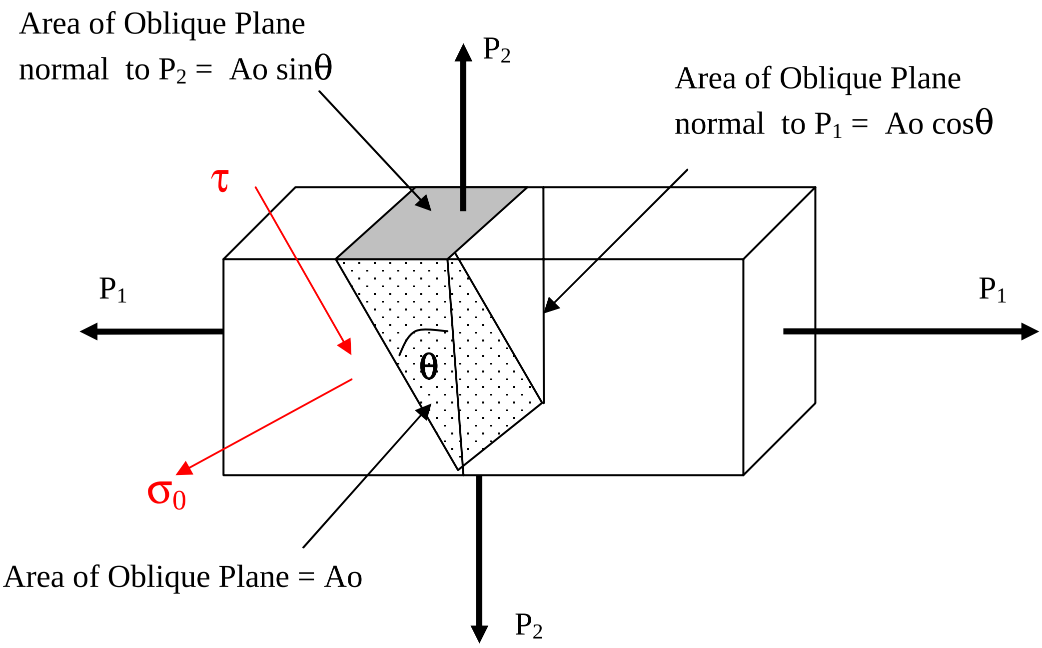
Note that this means that **the shear stress will be at its maximum** value when sin 2θ, is equal to its maximum value of 1, which will be **when θ is at 45^0**. Further it means that:

$\tau_{MAX} = ½\, \sigma \times 1$, which means that :-

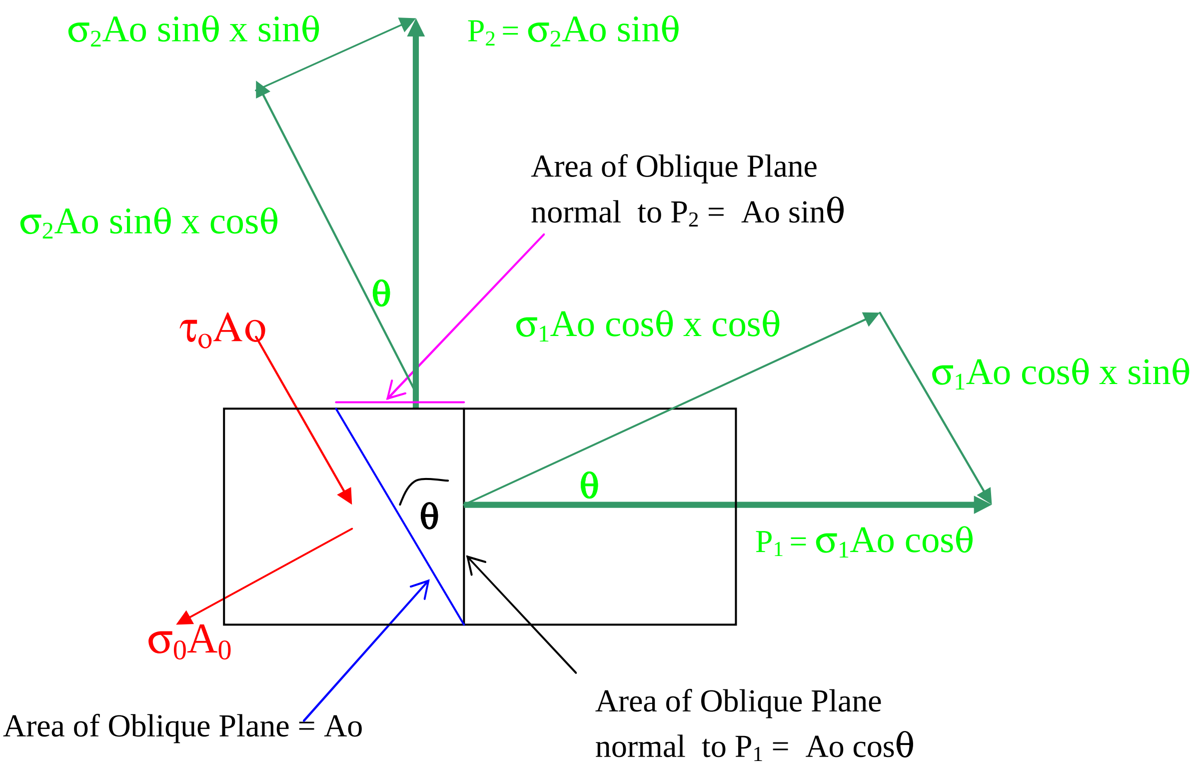
The maximum shear stress is equal to <u>half</u> the stress on the plane normal to the applied force.

Mutually Perpendicular Forces Acting On An Oblique Plane.

So far we have considered a load acting in one direction only. Suppose now that the body is subjected to mutually perpendicular loads P_1 and P_2 as shown.



Considering now the FORCES at work:



Equating Forces normal to the plane,

$$\sigma_0 A_0 \quad = \quad \sigma_1 Ao\cos\theta \times \cos\theta + \sigma_2 Ao\sin\theta \times \sin\theta$$

$$\sigma_0 \quad = \quad \underline{\sigma_1\cos^2\theta + \sigma_2\sin^2\theta}$$

Equating Forces parallel to the plane,

$$\tau Ao \quad = \quad \sigma_2 Ao\sin\theta \times \cos\theta - \sigma_1 Ao\cos\theta \times \sin\theta$$

$$\tau \quad = \quad \underline{(\sigma_2 - \sigma_1)\sin\theta\cos\theta}$$

Example 1

A welded air receiver has a working pressure of 30 bar. The dimensions of the receiver are 2.2 metre outside diameter and 30 mm thick.

Calculate:

(a) the tensile stress on the main longitudinal weld;
(b) the tensile and shear stress on a repair weld at 40^0 to the main longitudinal weld.

Solution

You should remember the "thin cylinder" formulae from previous (Class Two) work:

Hoop Stress σ_{HOOP} = pd/2t $= \dfrac{30\times10^2(2.2-0.06)}{2 \times 0.03} =$ **$\underline{107 MN/m^2}$** **ANS a)**

Where p = pressure (N/m^2)
d = diameter (m)
t = thickness (m)

Similarly,

Circumferential Stress σ_{CIRC} = pd/4t = **$\underline{53.5 MN/m^2}$**

As usual, the stress on the longitudinal joint is twice that on the circumferential joint.

For part b) we need to apply the formulae we have just derived.

$$\sigma_0 \quad = \quad \sigma_1\cos^2\theta + \sigma_2\sin^2\theta$$

$$\therefore \sigma_0 = \sigma_{HOOP} \cos^2\theta + \sigma_{CIRC} \sin^2\theta$$

$$\therefore \sigma_0 = 107\cos^2 40^o + 53.5 \sin^2 40^0$$

$$\therefore \sigma_0 = \underline{\mathbf{84.89MN/m^2 \ \ ANS\ b)}}$$

For the shear stress, we have that

$$\tau = (\sigma_2 - \sigma_1) \sin\theta \cos\theta$$

$$\therefore \tau = (\sigma_H - \sigma_C) \sin\theta \cos\theta$$

$$\therefore \tau = (53.5 \text{ MN/m}^2) \sin 40 \cos 40$$

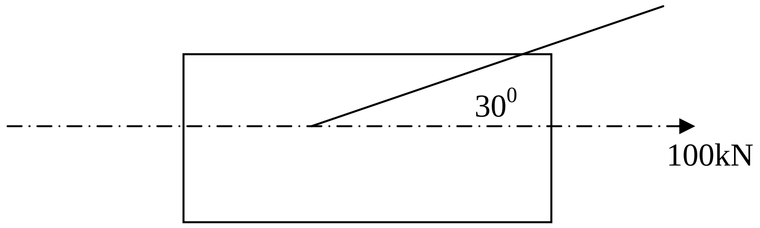
$$\therefore \tau = \underline{\mathbf{26.34\ MN/m^2 \quad ANS\ b)}}$$

Example 2

A thin walled cylindrical vessel of internal diameter 300mm and wall thickness 3mm is subjected to an axial tensile load of 100 kN, and an internal pressure of 35 bar.

Calculate each of the following:

(a) the normal and shear stress on a plane of 30^o to the axial load.
(b) The resultant combined stress and the angle of this stress to the plane of the axial load



$\sigma x = \sigma hoop = pd/2t = 35x10^5 x0.3/2x0.003 = 175MN/m^2$

$\sigma y = \sigma circumf. + \sigma tensile = pd/4t + 100x10^3/\pi \text{ x } 0.3 \text{ x } 0.003$

$\therefore \sigma y = 175/2 + 35.38 = 122.9 \text{ MN/m}^2$

$$\sigma_0 = \sigma_1 \cos^2\theta + \sigma_2 \sin^2\theta$$

$$\therefore \sigma_0 = \sigma_X \cos^2\theta + \sigma_Y \sin^2\theta$$

$$\therefore \sigma_0 = 175\cos^2 30^o + 122.9 \sin^2 30^0$$

$$\therefore \sigma_0 = \underline{\mathbf{162\ MN/m^2\ \ ANS}}$$

$$\tau = (\sigma_2 - \sigma_1) \sin\theta \cos\theta$$

$$\therefore \tau = (\sigma_X - \sigma_Y) \sin\theta \cos\theta$$

$$\therefore \tau = (175 - 122.9) \sin 30 \cos 30$$

$$\therefore \tau = \underline{\mathbf{22.57\ MN/m^2\ \ ANS}}$$

Resultant = $\sqrt{\sigma_o^2 + \tau^2} = \sqrt{162^2 + 22.57^2}$ = **<u>163.5 MN/m^2 ANS</u>**

Angle of resultant = $\tan^{-1} (\sigma o/\tau) = \tan^{-1} (162/22.57) = 82.07^0$

$\therefore$ Angle = 82.07 + 30 = **<u>112.07^o to the longitudinal axis ANS</u>**

Self Assessed Questions for you to try. (Answers given)

1. A pressure vessel of welded construction is 800 mm diameter and the thickness of the shell material is 25 mm. The working pressure is 2.5 MN/m^2 and the circumferential seams are welded at 45^0. Determine the normal tensile and shear stresses at the weld.

   Ans: $\sigma_T = 30\ MN/m^2$ $\tau = 10\ MN/m^2$

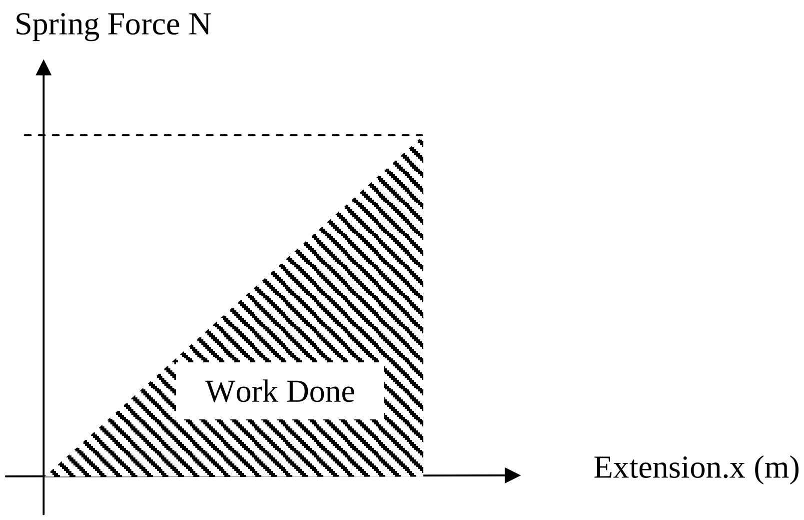
2. A thin shell cylinder has circumferential and longitudinal stresses of 100 MN/m^2 and 50 MN/m^2 respectively. Find the normal tensile and shear stresses on a section at 35^0 to the circumferential (hoop) stress. Determine also the resultant stress on the section and its direction relative to the normal tensile stress.

   Ans: $\sigma_T = 66.5\ MN/m^2$ $\tau = 23.5\ MN/m^2$ $\sigma_R = 70.5\ MN/m^2$ at 20^0 to σ_T.

ELASTIC STRAIN ENERGY AND IMPACT LOADING

All engineers should be familiar with the old rule that “a suddenly applied load is twice the gradually applied load”. In this section we take a simple approach to see why this is so, and discuss what happens when a suspended load is brought to rest.

In the section on Conservation of Energy we look at the concept of energy stored in a spring. If we plot a graph of Load v Extension for a spring then the area under the graph represents the work done.



This shows that the work done = ½ Px (or ½ kx^2), based on the fact that the average load on the spring is

$$\frac{0 + P}{2} = ½ P$$

However, if the load were applied **suddenly**, then the whole of the load would act all the time and the work done would be ***force x distance*** = Px. This in turn means that the energy stored would also be Px, that is **TWICE** the energy stored compared to a gradually applied load.

In fact the material under load does not need to be a spring. If we apply load to a metal within its elastic limit, then the metal behaves in a similar way to a spring, albeit that the amount of compression or extension is much smaller. The energy stored in a material strained by compression or extension is termed **Elastic Strain Energy** or **Resilience.**

Since Stress is defined as Load/Area, it follows that Load P = Stress (σ) x Area (A)

We also know that Strain ε = extension/length (x/L). It follows that x = ε x L

Also since $\dfrac{\text{Stress } \sigma}{\text{Strain } \varepsilon} = E$, (Young's Modulus of Elasticity), then

Extension $x = \dfrac{\text{stress x length}}{E} = \dfrac{\sigma L}{E}$

In order to find the strain energy stored in a body we equate it to the work done on the body.

Thus, Strain Energy = work done = ½ Px

$= \frac{1}{2} (\sigma \, x \, A) \, x \, (\sigma L/E)$

$= \dfrac{\sigma^2 \, x \, A \, x \, L}{2E}$

But Area x Length = Volume

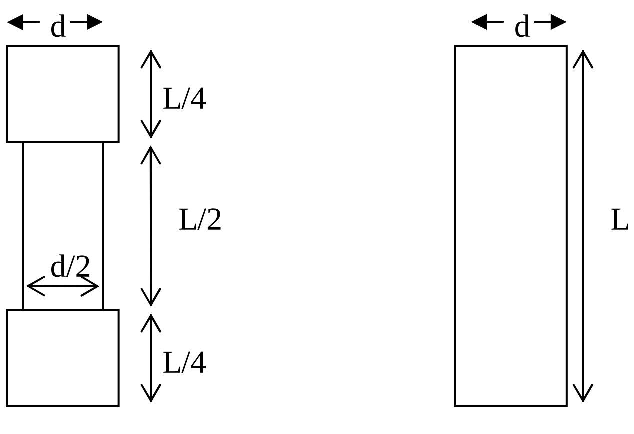
$\therefore$ Strain Energy $= \dfrac{\sigma^2 \, x \text{ volume}}{2E}$ →

This is a formula that needs to be remembered, but we should not forget the formula from which it came and that still applies, that Strain Energy = ½ Px.

A question sometimes asked in engineering knowledge exams is why "waisted" studs are used in some engineering applications. Part of the answer to this lies in the increased resilience stored by such a shape due to the narrowing of the middle section. Let us demonstrate this by means of an example.

Example 1

Compare the resilience of the two simplified shapes A and B representing waisted and un-waisted studs, made from the same material subject to an axial load 'P'.



A **B**

Resilience = Strain Energy Stored = $\dfrac{\sigma^2 \; x \text{ volume}}{2E}$

$$\text{Strain Energy Stored in 'B'} = \frac{\left(\dfrac{P}{\dfrac{\pi d^2}{4}}\right)^2 x \dfrac{\pi d^2}{4} xL}{2E}$$

$$\therefore \text{ Strain Energy Stored in 'B'} = \frac{2P^2 L}{\pi d^2 E}$$

The strain energy stored in 'A' will be the sum of the energy stored in its three sections, two of which are identical

$$\text{Strain Energy Stored in 'A'} = 2\frac{\left(\dfrac{P}{\dfrac{\pi d^2}{4}}\right)^2 x \dfrac{\pi d^2}{4} x \dfrac{L}{4}}{2E} + \frac{\left(\dfrac{P}{\dfrac{\pi d^2}{16}}\right)^2 x \dfrac{\pi d^2}{16} x \dfrac{L}{2}}{2E}$$

$$\text{Strain Energy Stored in 'A'} = \frac{4P^2 L}{4\pi d^2 E} + \frac{16P^2 L}{4\pi d^2 E} = \frac{5P^2 L}{\pi d^2 E}$$

So Energy Stored in A compared to Energy Stored in B = $\dfrac{5P^2L}{2P^2L}$

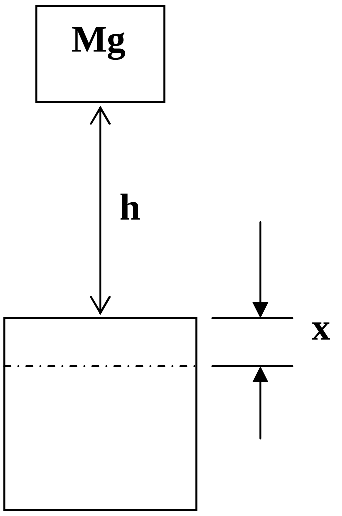
Which is 2.5 to 1, that is the waisted stud in this case is **two and a half times more resilient** than a stud of constant cross section. Of course, the waisted stud, being thinner in the mid section will also elongate more for the same tightening load.

Strain Energy due to falling loads

When a load falls through a height ‘h’ and, for instance, strikes a metal bar, then we must consider the potential energy of the load when calculating the strain energy stored in the bar. Note that we must include the additional compression of the bar, ‘x’ when calculating the potential energy.

First, let us put down what happens in words,

Potential Energy of load = Increased Strain Energy stored in the bar



$$\mathbf{Mg(h + x) = \frac{\sigma^2 \, x \, volume}{2E}}$$

Note that if the height ‘h’ is zero we get a familiar result:

$$Mgx = \frac{\sigma^2 \, x \text{ volume}}{2E}$$

But $\varepsilon = x/L$ so $x = L.\varepsilon = L\sigma/E$

$$\therefore \quad Mg \; x \; \frac{L\sigma}{E} = \frac{\sigma^2 \, x \, A \, x \, L}{2E}$$

$$\therefore \quad Mg = \sigma A/2$$

$$\therefore \quad \sigma = 2(Mg/A)$$

But (Mg/A) is the stress which would result if the load were applied gradually. So once again we have shown ***that the suddenly applied load is twice the gradually applied load.***

Example 1

A mass of 5 tonne is dropped 50mm onto a cast iron column of 80 mm diameter.

Calculate the minimum length of the column if the energy of impact is to be absorbed without raising the maximum instantaneous stress above 220 MN/m^2.
Take $E = 110 \times 10^9$ N/m^2

Solution

Potential Energy of load = Increased Strain Energy stored in the bar

$$\mathbf{Mg(h + x) = \frac{\sigma^2 \, x \, volume}{2E}}$$

But $\quad x = L.\varepsilon \quad = \quad L\sigma/E$

$$\therefore Mg(h + L\sigma/E) = \frac{\sigma^2 \, x \, A \, x \, L}{2E}$$

$$5000g\left(0.05+\frac{220\times10^6 L}{110\times10^9}\right)=\frac{(220\times10^6)^2\times\pi 0.04^2 L}{2\times110\times10^9}$$

$\therefore$ 0.05 + 0.002L = 0.0225L

$\therefore$ L **= 2.43m ANS**

Of course it is not just potential energy that may have to be considered when considering falling loads. A moving body will possess Kinetic Energy, and this must be taken into consideration when a moving body is brought to a halt. Let us illustrate this with another example.

Example 2

A cylinder cover of mass 2.4 tonne is hung from four steel wires of diameter 10mm. The cylinder cover is lowered to allow fitting, at a steady speed of 0.2 m/sec when the brake is suddenly applied. At this point the wires are 42 m long.

Calculate each of the following:

(a) the static stress on the wires when holding the cylinder cover
(b) the additional stress imposed on the wires due to the sudden stop
(c) the maximum extension of the wires due to the sudden stop

The Modulus of Elasticity of the steel wire material is 208 GN/m^2.

Solution

a) Direct Stress $\sigma_D = \dfrac{600 \ x\ 9.81}{\pi/4 \ x\ 0.01^2} = \underline{75 \text{ MN/m}^2}$ ANS a)

b) Potential Energy and Kinetic Energy of Load = Strain Energy on wires

Note, this is not a body falling from a height, so we are only concerned about the extension of the wires 'x' when considering potential Energy.

$$\therefore \text{Mg } (L\sigma/E) + ½ \text{ Mv}^2 = \frac{\sigma^2 \ x \ A \ x \ L}{2E}$$

$$\therefore \frac{2400 \ x \ 9.81 \ x \ 42 \ x \ \sigma}{208 \ x \ 10^9} + ½ \ x \ 2400 \ x \ 0.2^2 = \frac{\sigma^2 \ x \ (\pi/4) \ x \ 0.01^2 \ x \ 42}{2x \ 208 \ x \ 10^9}$$

$$\therefore 4.754 \ x \ 10^{-6}\sigma + 48 = 31.718 \ x \ 10^{-15}\sigma^2 + 48$$

$$\therefore \sigma^2 - 149.9 \ x \ 10^6 \ \sigma - 1.5133 \ x \ 10^{15} = 0$$

A quadratic equation from which we get that σ = 159.4 MN/m^2 ANS b)

c) Extension **due to the sudden stop**, $x = L\sigma/E$

$$\therefore x = \frac{159.4 \ x \ 10^6 \ x \ 42}{208 \ x \ 10^9} = 0.03218\text{m}$$ (32.18 mm) ANS c)

Self Assessed Questions for you to try. (Answers given)

1. A compound bar is formed from a solid steel rod of 20 cm^2 cross sectional area and 1m long shrink fitted with a bronze liner of the same length with a cross sectional area of 40 cm^2. The bar is arranged vertically with a collar at its lower end, onto which a mass of 1 tonne is dropped from a height of 100 mm. Find the maximum instantaneous stress induced in each material.

   Take E_{STEEL} = 200 GN/m^2, and E_{BRASS} = 200 GN/m^2

   Ans: σs = 315 MN/m^2, σb = 157 MN/m^2

2. A mass of 1 tonne suspended from a crane by a chain having a cross sectional area of 6 cm^2 is being lowered at a uniform speed of 0.6 m/sec. At the instant when the length of the chain unwound is 10 m the motion is suddenly stopped. Determine the maximum instantaneous stress and elongation produced in the chain due to this sudden stop. Neglect the weight of the chain and take E = 200 GN/m^2.

   Ans: 126 MN/m^2, 6.3 mm.