EULER’S THEORY FOR LONG SLENDER STRUTS

In the section on Combined Bending and Direct Stresses it was shown that an offset load from the neutral axis produces both direct and bending stresses. If the load were placed directly on the neutral axis then only a direct compressive (or tensile) stress would result.

However, this only applies if the component is considered to be very short in comparison to its cross section dimensions. The longer the length becomes, then the more likely the component is to fail by **BUCKLING**.

A strut is a member subjected to a direct compressive stress. The load carrying capacity of relatively short struts with large cross section area is limited by the crushing strength of the material. Long and slender struts, however, can become unstable and tend to buckle.

A small transverse load applied to the mid-point of a slender strut will produce a lateral deflection, which disappears when the transverse load is removed. As the compressive load is increased a point is reached at which the lateral deflection does not disappear. At this point the strut is in a state of unstable equilibrium and the slightest lateral disturbance will cause it to buckle. Such a strut has clearly reached the limit of its load carrying capacity, and the load is said to have reached its critical value.

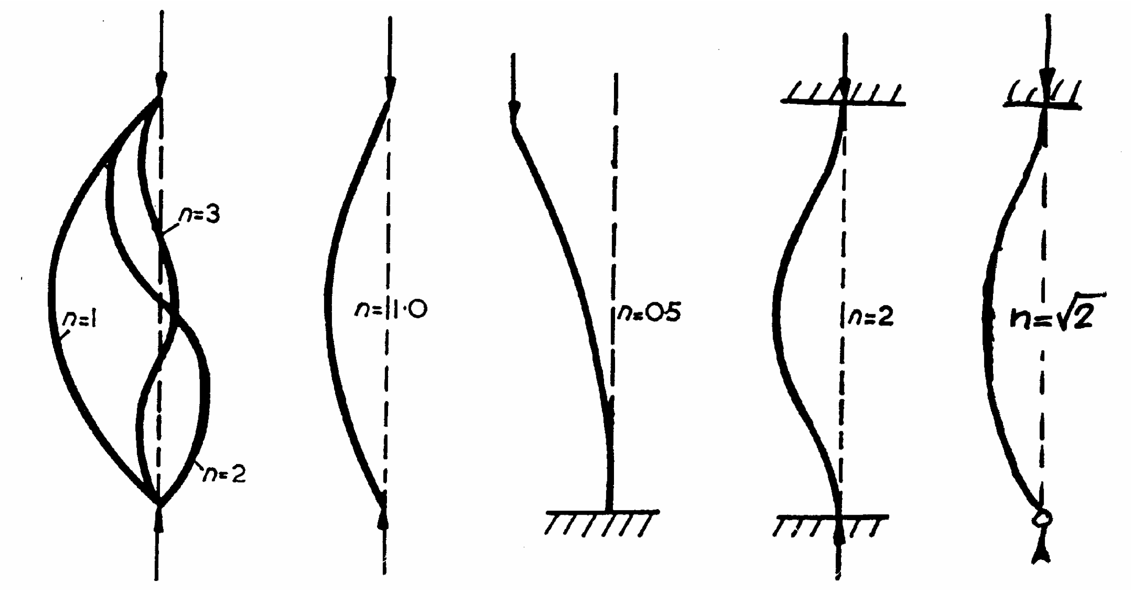
The critical load for a strut may be found using **EULER’S THEORY**, which is based on the assumptions below:

1. The material is homogenous.
2. The load is applied axially at the centroid of the section.
3. The cross section is uniform.
4. The strut is initially straight.
5. The direct stresses due to the compressive load are negligible compared with the bending stresses induced by buckling.

It can be shown that Euler’s Critical load, F_E Newtons, is given by:

$$\boxed{F_E = \frac{n^2\pi^2 EI}{L^2}}$$

F_E is the critical Euler load to cause buckling.
E is Young’s Modulus.
I is the MINIMUM second moment of area of the cross section of the strut.
L is the length of the strut.
n depends on the manner of buckling. This is a number which is determined from the way in which the strut is supported. There are numerous ways in which the strut can be supported, but the main ones are:



(a) (b) (c) (d) (e)

Diagram (a) shows a strut pin jointed at each end. There are several modes of failure as illustrated, but it is the smallest value of load, and therefore n, which will cause failure. In this case the failure will be when n = 1 as in diagram (b).

Diagram (c) shows a strut built in at one end and free to move at the other.

Diagram (d) shows a strut built in at both ends.

Diagram (e) shows a strut built in at one end and allowed to move only vertically at the other.

Each of the above diagrams shows that the strut will deflect into a complete or part sine wave. The value of n is the **NUMBER OF COMPLETE HALF SINE WAVES**. The larger the value of n (and hence the larger the number of half sine waves) the larger the load needed to cause buckling. If the strut can be supported along its length as often as possible so that no lateral movement is possible, the value of n increases and the strut becomes more stable.

Effective Length of a strut

If a strut is made to deflect into two half sine waves, then it could be considered as two separate struts which are in series, each being half the length of the original strut. The critical load will then be greater.

In general, if a strut is made to deflect into n half sine waves, then the strut becomes 'n struts' each of length $\frac{L}{n}$ where L is the length of the original strut. Therefore

$$\text{Effective Length } l = \frac{L}{n}$$

The Euler equation then becomes

$$F_E = \frac{\pi^2 EI}{l^2}$$

Slenderness Ratio

Since $I = Ak^2$ where A is the cross sectional area of the strut and k is the MINIMUM radius of gyration, then Euler's equation can be written as

$$F_E = \frac{n^2\pi^2 EAk^2}{L^2} = \frac{\pi^2 EAk^2}{l^2}$$

$$F_E = \frac{\pi^2 EA}{\left(\frac{l}{k}\right)^2}$$

This shows that for a given cross sectional area A, the critical load F_E is inversely proportional to the square of the ratio $\frac{l}{k}$. This determines when instability will start and is called the Slenderness Ratio. For most engineering materials and applications, Euler's equation can only be used when the slenderness ratio is greater than **120**. At values less than this the strut will be stable and will only suffer direct compressive stress.

WORKED EXAMPLE

A strut is 2 m long and has a rectangular cross section 30 mm × 20 mm. It is pin jointed at each end and is constrained to move axially in guides. E for the material is 200 GN/m^2.

(a) What is the value of n to be used in Euler's equation?

This strut is the same type as in diagram (b) in the notes.

Therefore the value of n is 1. Answer

(b) What is the effective length of the strut?

The effective length is $l = \frac{L}{n} = \frac{2}{1}$ = **2 meters** Answer

(c) What is the cross sectional area of the strut?

Area = 30 × 20 = **600 mm^2** Answer

(d) What is the minimum second moment of area I?

The value of I, for a rectangular cross section, is given by $I = \frac{bd^3}{12}$. Therefore the minimum will be when b = 30 mm and d = 20 mm.

$$I = \frac{bd^3}{12} = \frac{30 \times 20^3}{12} = \mathbf{20000\ mm^4} \text{ Answer}$$

(Note! The maximum value of I would be when b = 20 mm and d = 30 mm giving I as 45000 mm^4)

(e) What is the minimum radius of gyration, k?

$$k^2 = \frac{I}{A} = \frac{20000}{600} = 33.33 \text{ mm}$$

$$k = \sqrt{33.33} = \mathbf{5.774\ mm} \text{ Answer}$$

(f) What is the slenderness ratio for the strut?

$$\text{Slenderness ratio} = \frac{l}{k} = \frac{2000 \text{ mm}}{5.774 \text{ mm}} = \underline{\mathbf{346}} \text{ Answer}$$

This means that since the value is greater than 120, Euler's equation can be used to determine the critical load for buckling.

(g) Determine Euler's Critical Load for this strut.

$$F_E = \frac{\pi^2 EA}{\left(\frac{l}{k}\right)^2}$$

$$F_E = \frac{\pi^2 \times 200 \times 10^9 \times 600 \times 10^{-6}}{346^2}$$

$$F_E = \frac{1184 \times 10^6}{0.1197 \times 10^6}$$

$$F_E = 9891\text{N} = \underline{\mathbf{9.891\ kN}} \text{ Answer}$$

This means that if the load is less than 9.891 kN, the strut will be stable and will only suffer a direct compressive stress. If the load is greater than 9.891 kN, the strut will be unstable and will fail due to buckling.

STUDENT EXAMPLES

1. A vertical strut is 16 m long. Its cross section is a symmetrical I section where the thickness of material is 10 mm. The flanges are 250 mm long and the web is 300 mm long. The strut is built in at both ends and E for the material is 200 GN/m^2.
(a) Calculate the minimum value of I.
(b) Calculate the slenderness ratio.
(c) Calculate the critical load.

(a) 26.07×10^{-6} m^4 (b) 140 (c) 805.6 kN

2. A strut consists of a straight metal bar 1 m long and of rectangular cross section 12 mm × 5 mm. It is pin jointed at each end and E for the material is 70 GN/m^2.
(a) Calculate the minimum value of I.
(b) Calculate the slenderness ratio.
(c) Calculate the critical load.

(a) 0.125×10^{-9} m^4 (b) 693 (c) 86.36 N

3. An alloy tube of length 3.2 m has an external diameter of 18 mm and an internal diameter of 13 mm. When subjected to an axial tensile force of 4.5 kN, the extension of the tube was 1.1 mm.
(a) Calculate the modulus of elasticity for the tube material.
(b) The tube is to be used as a vertical strut to carry an axial compressive load.
 (i) If the ends are pin jointed, calculate Euler's critical load.
 (ii) If one end is built in and the other end free, calculate Euler's critical load.

(a) 107.5 GN/m^2 (b)(i) 388.8 N (ii) 777.6 N