

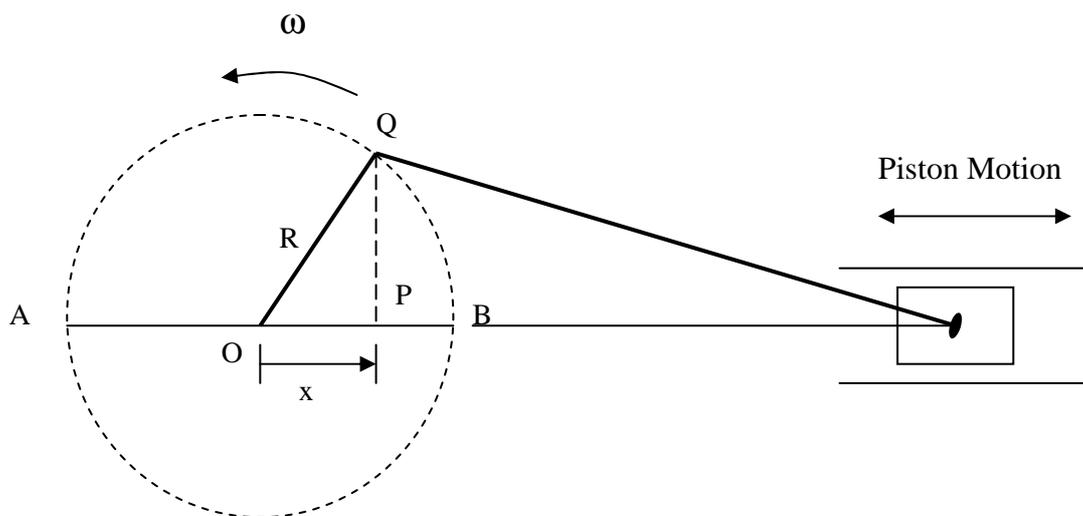
SIMPLE HARMONIC MOTION.

Periodic motion

The motion of a body or particle can be defined as "periodic" if it moves to and fro so that every part of its motion recurs regularly. Many devices in engineering oscillate at regular intervals. Simple Harmonic motion is a **special type** of periodic motion. Not everything which moves with Periodic Motion moves with Simple Harmonic Motion. In order for the motion to be classed as Simple Harmonic then the following conditions must be satisfied:

1. The acceleration is always directed towards a fixed point.
2. The acceleration is proportional to its distance from that fixed point. *Note that this implies the acceleration is constantly changing and therefore the equations used for uniform acceleration do NOT apply.*

Simple Harmonic Motion can be generated from circular motion, and the method used here for demonstrating S.H.M and solving S.H.M. problems, is to use an auxiliary circle and a generating vector. One familiar object that moves with S.H.M. is the piston of a reciprocating engine. It is perhaps worth starting with this analogy to help us understand what is happening with S.H.M. The crankshaft moves with constant angular velocity ω , but the piston moves back and forwards along the cylinder with S.H.M. The displacement of the piston can be found by projecting a point 'P' down from point 'Q' on the crank web O.Q.



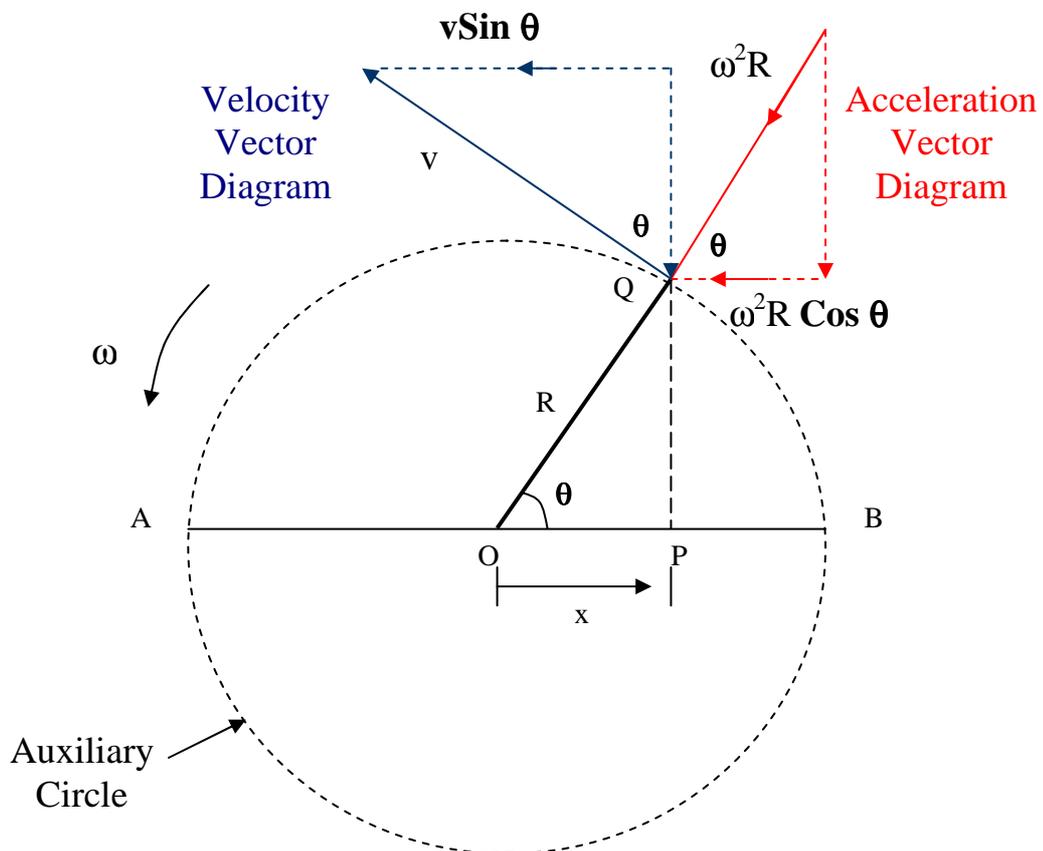
Derivation of Simple Harmonic Motion

We will come across other objects that move with S.H.M. where there is no physical object moving in a circle, such as the simple pendulum, but we can still use the auxiliary circle method as a mathematical concept to help us understand the motion and solve problems. So let us dispense of the con-rod and piston and focus our attention on the rotating vector "OQ" and the displacement of point P.

Point "Q" moves in a circular path of radius "R" at constant angular velocity " ω ".

Point "P" is drawn as the projection of point Q on the diameter AB of the auxiliary circle.

It is the motion of the point P along AB that will be seen to be Simple Harmonic.



From previous work, the instantaneous linear velocity of point Q will be tangential to the radius and of magnitude ωR . The instantaneous linear acceleration will be the centripetal acceleration, $\omega^2 R$, radially inwards. We can therefore take components of these vectors to find the instantaneous velocity and acceleration of point P along the line AB.

INSTANTANEOUS VELOCITY

Referring to the velocity vector diagram:-

$$\begin{aligned}\text{Instantaneous velocity of P along AB} &= v \sin \theta \\ &= \omega R \sin \theta\end{aligned}$$

$$\text{From Triangle O.P.Q:-} \quad PQ = R \sin \theta$$

$$\text{also} \quad PQ = \sqrt{R^2 - X^2}$$

$$\text{HENCE:- Instantaneous Velocity} = \underline{\omega \sqrt{R^2 - X^2} \text{ m/s}} \rightarrow$$

You will need to remember this formula. It will enable us to calculate the instantaneous velocity of an object moving with S.H.M. at any displacement from the mid-point, 'x'. Note also that the velocity will reach a maximum its' maximum of ' ωR ' when $x = 0$, that is at the mid point. In relation to a piston this would be at mid stroke. It further confirms that at each end of the stroke, when $x = R$, the velocity is zero, i.e. the piston stops!

INSTANTANEOUS ACCELERATION

From the acceleration vector diagram:-

$$\text{Instantaneous acceleration of P along AB} = \omega^2 R \cos \theta$$

$$\text{From Triangle O.P.Q. :-} \quad x = R \cos \theta$$

$$\text{HENCE:- Instantaneous acceleration} = \omega^2 x \text{ m/s}^2$$

You will need to remember this formula Note that this means that **the acceleration of P is proportional to X and is always directed towards point 'O'. i.e. we have simple harmonic motion.** In relation to a piston this means that the acceleration is maximum at the maximum value of 'x', i.e. at the extremities of the stroke.

Terms and Definitions used in Simple Harmonic Motion

We will need to interpret the information given in a question in terms of the formulae developed from the auxiliary circle. A definition of some common terms will therefore be useful before we proceed.

Travel

Is the distance between extreme ends of the motion. This is directly analogous to the stroke of a diesel engine. Remember that we will travel two strokes in each revolution, and that the stroke is twice the distance from mid stroke to the end of the stroke, i.e. ***Travel* = 2 x R**

Displacement

Is the distance **measured from mid-travel** "X". In exam questions distances are often given from Top Dead Centre, and we must therefore convert this into the distance from Mid-stroke.

Amplitude

Is the maximum displacement from mid-travel
i.e. Amplitude = R (= half travel)

Periodic Time τ (tau)

Is the time required for one complete oscillation or vibration. It is also the time taken to make two strokes and the time to make one revolution

$$\text{Time for one revolution} = \frac{\text{angular displacement}}{\text{angular velocity}}$$

$$\text{or } \tau = \frac{2\pi \text{ (rad)}}{\omega \text{ (rad/sec)}} \text{ sec} \dots \dots \dots 1$$

But from:- $\text{Acceleration} = \omega^2 x$

$$\omega = \sqrt{\frac{\text{Acceleration}}{\text{Displacement}}}$$

Substituting for w in 1 gives seconds

$$\tau = \frac{2\pi}{\sqrt{a/x}} = 2\pi \sqrt{x/a}$$

Frequency (f)

Is the number of oscillations per second, units are (Hertz [Hz]).

$$\text{Frequency} = 1/\tau = \frac{1}{2\pi} \sqrt{x/a}$$

It may be best not to try and remember this equation, as it can be confused with the equation for τ . If we can remember the equation for τ and that $f = 1/\tau$ we will be able to find f .

Example 1

A body moves with simple harmonic motion and has a velocity of 12 m/s when the displacement is 50 mm from the origin and a velocity of 3 m/s when the displacement is 100 mm from the origin, calculate:-

- (a) The frequency of the oscillations
- (b) The amplitude of the oscillations
- (c) The acceleration when the displacement is 75 mm.

Solution

Instantaneous Velocity $v = \omega \sqrt{R^2 - x^2}$ m/s

So $12 = \omega \sqrt{R^2 - 0.05^2}$ from first condition and1

$3 = \omega \sqrt{R^2 - 0.1^2}$ from second condition2

Combining these equations, by dividing the first by the second gives:

$$\frac{12}{3} = \frac{\omega \sqrt{R^2 - 0.05^2}}{\omega \sqrt{R^2 - 0.1^2}}$$

Squaring both sides,

$$16 = \frac{R^2 - 0.05^2}{R^2 - 0.1^2}$$

Which gives $R = 0.1025\text{m}$ (The amplitude of the oscillations, ANS b)
 \rightarrow

Substituting for R in equation 1.

$$12 = \omega \sqrt{0.1025^2 - 0.05^2}$$

This gives $\omega = 134$ rads/sec.

$$F = \omega/2\pi = 21.3\text{Hz.}$$
$$\rightarrow$$

$$\text{At } x = 0.075\text{m, } a = \omega^2 x = 134^2 \times 0.075 = 1347 \text{ m/s}^2$$
$$\rightarrow$$

Example 2

A component oscillates harmonically between extremes 1.22 m apart, the periodic time being 2 seconds. Determine:-

- The angular velocity of the generating vector
- The maximum acceleration
- The velocity at a point 152 mm from the centre of oscillation
- Time taken to travel a distance of 76 mm from an extremity

Solution

a) Periodic time $\tau = 2\pi/\omega \therefore \omega = 2\pi/\tau = 2\pi/2 = \pi$ rads/sec.
 \rightarrow

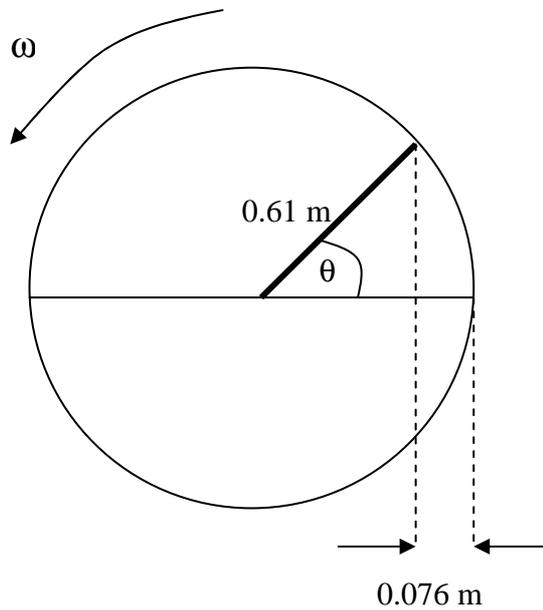
b) Maximum acceleration at maximum value of $x = R$.

$$a_{\max} = \omega^2 R = \pi^2 \times 0.61 = 6.02 \text{ m/s}^2$$
$$\rightarrow$$

c) velocity = $\omega \sqrt{R^2 - x^2}$

$$\therefore v = \pi \sqrt{0.61^2 - 0.152^2} = 1.856 \text{ m/sec}$$

d) To determine the time taken to travel a distance of 76mm from an extremity, we need to sketch the generating vector and auxiliary circle.



At $x = 0.61 - 0.076$ then $\cos \theta = \frac{0.61 - 0.076}{0.61}$

$\therefore \theta = 28.9^\circ$ or 0.504 rads.

And since the **generating vector** is moving at constant velocity, then

$$\theta = \omega t \text{ and } t = \frac{0.504}{\pi} = \underline{\underline{0.1605 \text{ sec}}}$$

Example 3

A certain machine part with a stroke of 250 moves with shm and makes 150 oscillations per minute. If the component has a mass of 60 kg, find:

- (a) the accelerating force acting on it when 75 mm from mid-stroke position;
- (b) the maximum accelerating force.

$$\text{Frequency} = f = \frac{150}{60} = 2.5 \text{ Hz.}$$

$$\omega = f \times 2\pi = 2.5 \times 2\pi = 15.71 \text{ rad/sec}$$

$$\text{from } a = \omega^2 \cdot x = (15.71)^2 \times 0.075$$

$$a = 18.51 \text{ m/s}^2$$

$$\text{accel. force} = \text{Mass} \times \text{acceleration} = 60 \times 18.51$$

$$\text{force} = 1.11 \text{ kN.}$$

Maximum acceleration is at $x = R$ and $R = 0.125 \text{ m}$

$$a_{\text{max}} = \omega^2 \cdot R = (15.71)^2 \times 0.125$$

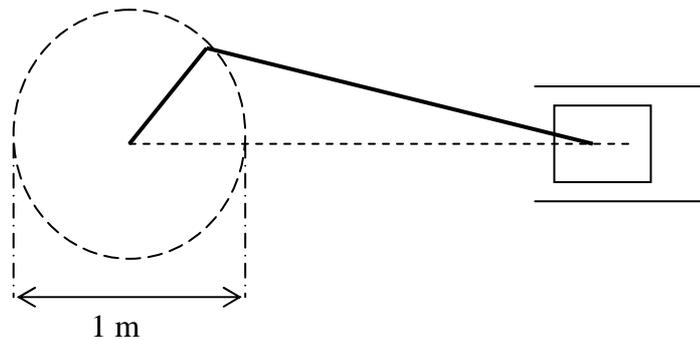
$$= 30.85 \text{ m/s}^2$$

$$\text{Max. accel. force} = \text{Mass} \times \text{acceleration} = 60 \times 30.85$$

$$= 1.851 \text{ kN.}$$

Example 4

The piston of a horizontal reciprocating engine moves with shm. The stroke of the engine is 1m and its speed 240 rev/min. If the piston has a mass of 20 kg, determine the driving force required to overcome its inertia at the beginning and end of stroke and also at $\frac{1}{4}$, $\frac{1}{2}$ and $\frac{3}{4}$ travel.



$$\omega = \frac{2\pi \times 240}{60} = 25.13 \text{ rad/sec}$$

$$x_{\max} = 0.5 \text{ m} \text{ hence } a_{\max} = \omega^2 \cdot x_{\max}$$

$$a_{\max} = 25.13^2 \times 0.5 = 315.8 \text{ m/s}^2$$

The maximum acceleration is at the beginning and end of the stroke:-

$$\text{accel. force} = \text{mass} \times \text{acceleration} = 20 \times 315.8 = 6.316 \text{ kN.}$$

at 1/2 travel $x = \text{zero}$ so acceleration = zero and accel. force = zero

at 1/4 & 3/4 travel $x = 0.25$

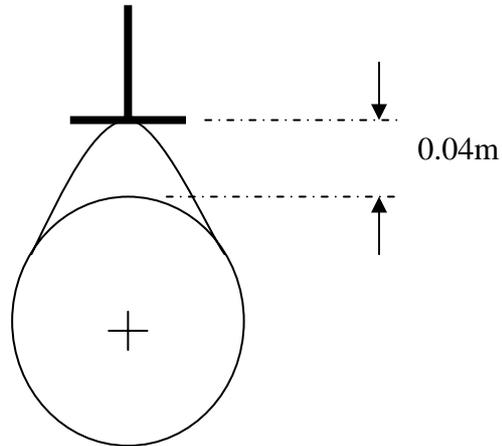
$$a = \omega^2 \cdot x = 25.13^2 \times 0.25 = 157.9 \text{ m/s}^2$$

$$\text{Thus Force} = ma = 20 \times 157.9 = 3158 \text{ N}$$

Example 5

A cam is to lift a follower through a distance of 40 mm with simple harmonic motion. The cam rotates at 180 rev/min and operation of the follower is carried out in 1/4 of a revolution of the cam. Find the maximum velocity and acceleration of the follower.

Solution



Time for one Rev. = 1/3 sec. Hence:- For the follower the periodic time

$$\tau = \frac{1}{3} \times \frac{1}{4} = \frac{1}{12} \text{ sec.}$$

$$\omega = 2\pi \times 12 = 75.4 \text{ rad/sec}$$

$$x_{\max} = R = \frac{0.04}{2} = 0.02$$

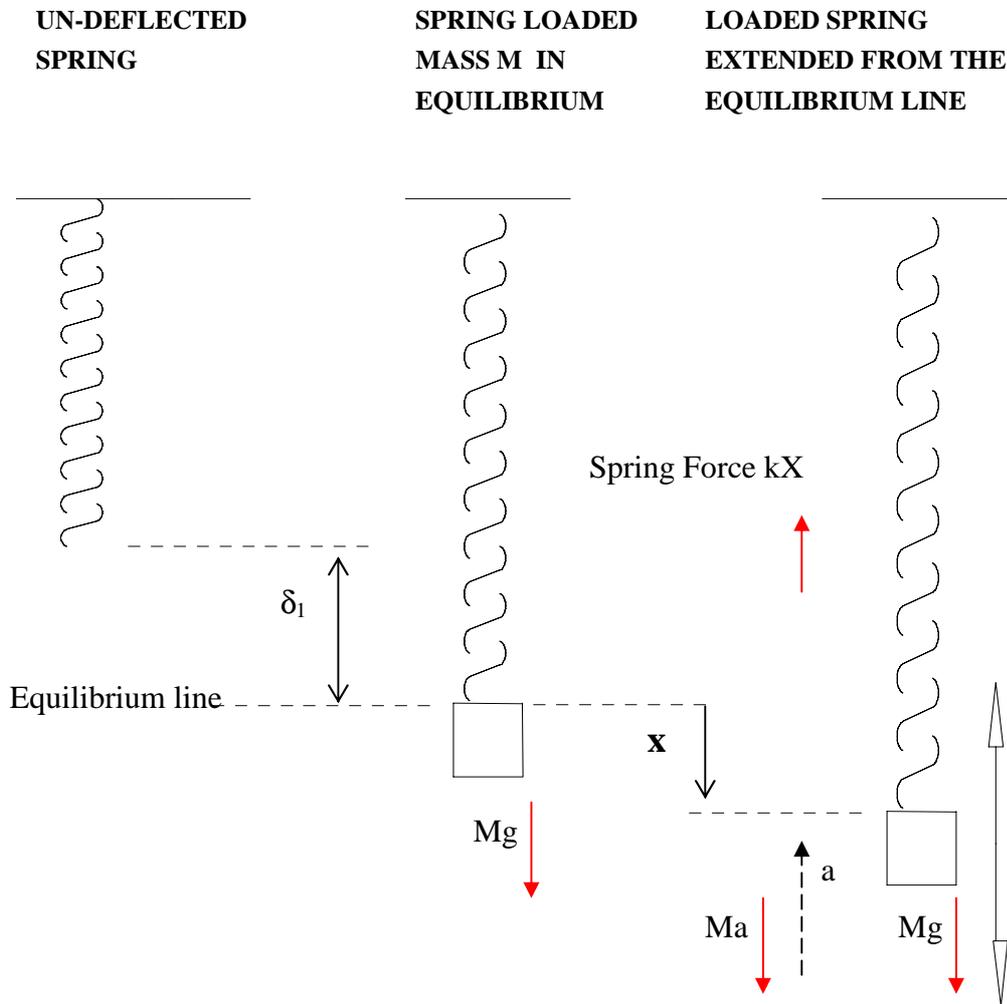
$$v_{\max} = \omega \sqrt{R^2} = 75.4 \times 0.02$$

$$= 1.508 \text{ m/s}$$

$$a_{\max} = \omega^2 \cdot R = 75.4^2 \times 0.02 = 113.7 \text{ m/s}^2$$

THE SPRING-MASS SYSTEM

Consider a spring of stiffness "k", supporting a mass "M"



We can show that when deflected from the equilibrium or "e line" by an external force, removal of that force will result in a vibration about the e-line.

At the instant of release the forces acting are:-

The spring restoring force = $k(\delta_i + X)$ acting upwards

The gravitational force = Mg acting downwards

These forces are of in the opposite direction, so the net restoring force is given by:-

$$\begin{aligned} \text{Restoring force} &= - Mg + k(\delta_i + X) = - Mg + k\delta_i + kX \\ &= kX \end{aligned}$$

This restoring force is equal to the mass \times acceleration:-

$$a = \frac{k X}{M}$$

i.e. The **acceleration is proportional to displacement from e-line and is always directed towards it**, which is the definition of simple harmonic motion.

Hence motion of the mass is simple harmonic

from:-

$$\tau = 2\pi \sqrt{\frac{\text{displacement}}{\text{acceleration}}} \qquad \tau = 2\pi \sqrt{\frac{X}{\frac{kX}{M}}} \qquad \tau = 2\pi \sqrt{\frac{M}{k}}$$

This is another equation we need to remember.

The foregoing neglected the mass of the spring. This is not really practical and it is sometimes necessary to make an allowance for the mass of the spring.

It can be shown that this allowance should be one third of the spring mass acting at the end of the spring.

This modifies the expression for periodic time to:-

$$\tau = 2\pi \sqrt{\frac{M + \frac{m}{3}}{k}}$$

Where M is the load mass and m is the total mass of the spring

From:-

$$\tau = 2\pi \sqrt{\frac{M}{k}}$$

The frequency f is given by:-

$$f = \frac{1}{\tau} = \frac{1}{2\pi} \sqrt{\frac{k}{M}}$$

also:-

$$f = \frac{1}{2\pi} \sqrt{\frac{\text{acceleration}}{\text{displacement}}}$$

Hence:-

$$\sqrt{\frac{k}{M}} = \sqrt{\frac{\text{acceleration}}{\text{displacement}}}$$

i.e.

$$\frac{\text{stiffness}}{\text{mass}} = \frac{\text{acceleration}}{\text{displacement}}$$

Example 6

A mass of 2 kg is placed on top of a vertical mounted spring of negligible mass having a stiffness of 1.5 kN/m, and a force applied such that the total deflection of the spring is 50 mm. If the external force is instantaneously removed, determine:

- (a) the maximum vertical height attained by the mass
(using the original position of the mass as a datum)
- (b) the maximum velocity of the mass.

Solution.

Firstly, we need to establish where the equilibrium line is. This will be the position of the mass when stationary under the action of its' own weight. For this condition,

$$Mg = kX$$

$$\therefore X = \frac{2 \times 9.81}{1.5 \times 10^3} = 13.08 \text{ mm}$$

This means the spring has been extended $(50 - 13.08) = 36.92$ mm from the equilibrium line. This gives us the value for the amplitude of vibration, the generating vector radius "R". It follows that it will move up a distance of 36.92 mm from the equilibrium line, which will be $(36.92 - 13.08) = \underline{23.84}$ mm higher than its original, unloaded position.

b) Maximum velocity occurs at mid-stroke when $X = 0$, and so

$$V_{\max} = \omega \sqrt{R^2 - X^2} = \omega R$$

$$\tau = 2\pi \sqrt{\frac{M}{k}}$$

$$\tau = 2\pi \sqrt{\frac{2}{1500}} = 0.2294 \text{ secs}$$

$$\omega = 2\pi/\tau = 27.39 \text{ rad/sec}$$

$$V_{\max} = 27.39 \times 0.03692 = \underline{1.01 \text{ m/sec}}$$

Example 7 (Exam Standard)

Two masses of 4 kg and 6 kg are connected to the end of a vertical spring of stiffness 6 kN/m. When the 4 kg mass is removed, and assuming simple harmonic motion,

calculate:

- the amplitude of vibration
- the frequency of vibration
- the linear velocity when the mass is 2 mm from the top of its travel
- the maximum kinetic energy of the mass

Solution

Extension of the spring with the 10kg load = $X_1 = F/k = 10g/6 \times 10^3 = 16.35\text{mm}$

Extension of the spring when 4kg load removed = $X_2 = F/k = 6g/6 \times 10^3 = 9.81\text{mm}$

This makes the amplitude of vibration $16.35 - 9.81 = \underline{6.54\text{mm ANS a)}$. This means the mass will move up and down 6.54mm from the equilibrium position of 9.81mm downwards from its free (totally un-loaded) length.

$$\tau = 2\pi \sqrt{\frac{M}{k}}$$

$$\tau = 2\pi \sqrt{\frac{6}{6000}} = 0.1987 \text{ secs} \quad \text{and Frequency} = 1/\tau = \underline{5.033 \text{ Hz ANS b)}$$

$$\omega = 2\pi/\tau = 31.62 \text{ rad/sec}$$

- c) 2mm from the top of travel, $X = 6.54 - 2 = 4.54\text{mm}$

$$V = \omega \sqrt{R^2 - X^2} = 31.62 \sqrt{0.00654^2 - 0.00454^2} = \underline{0.149 \text{ m/sec ANS c)}$$

- d) $V_{\max} = \omega \sqrt{R^2 - X^2}$ at mid point $X = 0 = \omega R$

$$V_{\max} = 31.62 \times 0.00654 = 0.207 \text{ m/sec}$$

$$\text{Kinetic Energy} = \frac{1}{2} Mv^2 = \frac{1}{2} \times 6 \times 0.207^2 = \underline{0.128 \text{ J ANS d)}$$

THE SIMPLE PENDULUM

The Simple Pendulum (Not to be confused with the conical pendulum discussed in connection with governors!) consists of a concentrated mass M suspended from a fixed point by a light cord. It can be shown that for small angles of swing that the simple pendulum moves with SHM. The method used to show this will be that used previously. We will **derive an expression for acceleration and see if it satisfies the criteria for SHM.**

Let length of cord = L and let mass be displaced so that cord moves an angle θ from the vertical. Linear distance from vertical = X [m]

When released the mass will oscillate in the vertical plane.

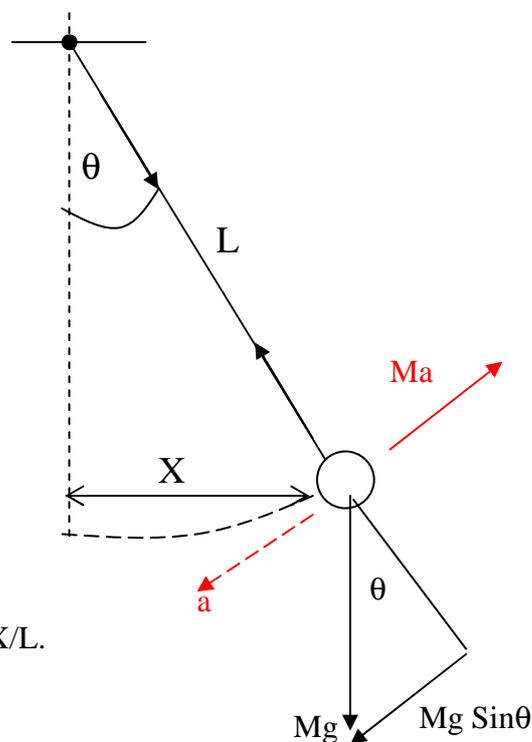
At instant of release, summing forces Normal to the chord:-

$$M \cdot a = Mg \sin \theta$$

$$a = g \sin \theta$$

For small angles, displacement of mass = X (approximates to length of arc) and $\sin \theta = X/L$.

$$a = \frac{gX}{L}$$



Hence acceleration is proportional to displacement and is directed towards an origin and is therefore simple harmonic.

$$\text{Since } \tau = 2\pi \sqrt{\frac{X}{a}} \text{ then } \tau = 2\pi \sqrt{\frac{X}{gX/L}} = 2\pi \sqrt{\frac{L}{g}}$$

This and the expression above for acceleration give us two more formulae to remember.

Example 8

Calculate the length of a simple pendulum which will make 1 oscillation in 2 seconds.

Solution

Periodic time t is 2 secs.

$$\text{From } \tau = 2\pi \sqrt{L/g}$$

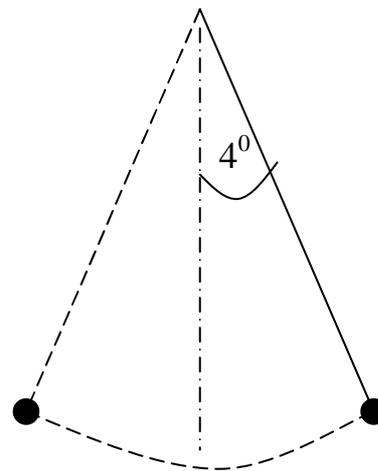
$$\text{Then } L = (\tau/2\pi)^2 \times g = \underline{0.994\text{m.}} \rightarrow$$

Hence the length of a pendulum on a grandfather clock!

Example 9

A simple pendulum was observed to perform 40 oscillations in 100 seconds, the angular amplitude of each being 4° , determine:-

- (a) The maximum linear acceleration of the mass
- (b) The length of the pendulum
- (c) The maximum velocity of the bob mass.



Solution

$$\text{Periodic time} = 100/40 = 2.5\text{secs.}$$

$$\text{From } \tau = 2\pi \sqrt{L/g}$$

$$\text{Then } L = (2.5/2\pi)^2 \times g = \underline{1.55\text{m. ANS b)}} \rightarrow$$

$$\text{Angular velocity of generating vector } \omega = \frac{\text{Distance in one revolution}}{\text{Time for one revolution}} = \frac{2\pi}{\tau} = 2.513 \text{ rads/sec}$$

$$\text{Maximum displacement} = \text{amplitude } R \cong L \cdot \sin\theta = 1.55 \sin 4^\circ = 0.1086\text{m}$$

$$\text{Maximum acceleration} = \omega^2 X_{\text{max}} = \omega^2 R = \underline{0.686 \text{ m/s}^2 \text{ ANS a)}} \rightarrow$$

$$\text{Maximum velocity occurs at } X = 0, = \omega R = \underline{0.273 \text{ m/sec ANS c)}} \rightarrow$$

Example 10

An open ended U-tube with vertical limbs, is partially filled with liquid and the level in each limb is the same. Show that if the level is depressed in one limb and then released, the liquid

column moves with S.H.M with a periodic time of $2\pi\sqrt{\frac{L}{2g}}$

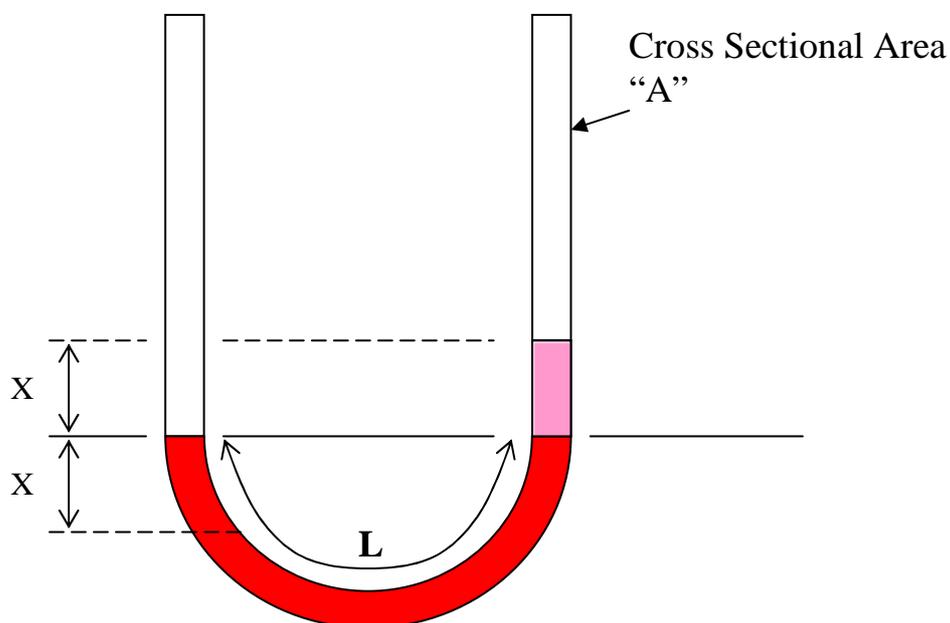
where L is the length of the liquid column in m.

Given L is 800 mm, find the frequency of oscillation and maximum velocity of the liquid column if the depression was 50 mm.

Solution

If we are to prove that the liquid moves with SHM then the method used to show this will be that used previously. We will **derive an expression for acceleration and see if it satisfies the criteria for SHM.**

First, draw a sketch:



Going back to first principles, from Newton, $F = m.a.$

Once the liquid has been depressed, then the restoring force is due to hydrostatic pressure.

$$F = \rho g A h, \text{ and here 'h' } = 2X, \quad \text{So } F = \rho g A 2X \quad \dots\dots\dots 1$$

$$\text{Mass of liquid} = \text{density} \times \text{Volume} = \rho A L \quad \dots\dots\dots 2$$

So combining these two equations with $F = m.a$,

$$\rho g A 2X = \rho A L a$$

Which gives $a = 2gX/L$

So the ***acceleration is proportional to the distance 'X' from the fixed point and directed towards it.*** We have Simple Harmonic Motion. Now we have proved we have SHM, we can use the general formula for S.H.M., in particular:

$$\tau = 2\pi \sqrt{\frac{X}{a}}$$

Which given our expression for 'a' gives:

$$\tau = 2\pi \sqrt{\frac{XL}{2gX}} = 2\pi \sqrt{\frac{L}{2g}} \quad \dots\dots\dots \text{Q.E.D.}$$

Given $L = 0.8\text{m}$ and $X = 0.05\text{m}$

$$\tau = 2\pi \sqrt{\frac{0.8}{2g}} = 1.269 \text{ secs}$$

$$\text{and } f = 1/\tau = \underline{0.788 \text{ Hz}} \quad \text{ANS}$$

Velocity of generating vector, $\omega = 2\pi/\tau = 4.952 \text{ rad/sec}$

Maximum velocity is at mid point ($X = 0$), and the amplitude $R = X_{\text{max}}$.

This gives max velocity = $\omega R = 4.952 \times 0.05 = \underline{0.248 \text{ m/s}} \quad \text{ANS}$

Self Assessed Questions for you to try. (Answers given)

1. The piston of a pump has a stroke of 300 mm and moves with simple harmonic motion.
- (a) If the pump performs 30 double strokes per minute, find the maximum velocity and acceleration of the piston.
 - (b) Given the mass of the piston as 100 kg, determine the accelerating force on the piston and its kinetic energy at the quarter stroke position.

Ans: 0.4713 m/s, 1.4805 m/s², 74.02 N, 8.3 J

2. An element of a machine has mass 80 kg and moves with shm through a stroke of 500 mm whilst performing 150 oscillations in a time of 1 min 30 s. Calculate:
- (a) the time taken for the element to move 150 mm from the beginning of its movement.
 - (b) the accelerating force at this position.

Ans: (a) 0.1107 s (b) 877 N

3. A camshaft rotating at 240 rev/min gives shm to a valve with a mass of 2 kg. The vertical lift of the valve is 60 mm and it is opened and closed in 110° movement of the camshaft. Determine the maximum force exerted between cam and valve.

Ans: 425 N

4. (a) A helical spring extends a distance of 80 mm when a force of 2 kN is applied to it. If a mass of 100 kg is now attached to its free end, pulled down and then released, determine the frequency of the oscillation.
- (b) If the maximum deflection was 50 mm, find the velocity and acceleration when 30 mm from the equilibrium position.

Ans: (a) 2.52 osc/s (b) 0.63 m/s and 7.5 m/s²

5. A helical spring with a mass of 6 kg has a mass of 10 kg attached to its free end. This mass is now pulled down 40 mm from the equilibrium position and then released. If the stiffness of the spring is 532 N/m, find the frequency of oscillation.
- Determine the kinetic energy of the mass when 30 mm from the equilibrium position.

Ans: (a) 1.06 osc/s (b) 0.155 J

- 6 A body moves with shm such that at 75 mm displacement from mid-travel, its velocity is 2 m/s and at 125 mm displacement, its velocity is 1.5 m/s. Calculate:
- (a) the amplitude of its motion;
 - (b) the periodic time.

Ans: (a) 0.169 m (b) 0.475 s

- 7 A valve on a petrol engine lifted vertically by a cam moves with shm. The lift of the valve is 5 mm and it is opened and closed in one quarter of a revolution of the cam. The mass of the valve is 100 gramme, and at its highest and lowest positions it is subjected to a spring force of 900 N and 180 N respectively. If the camshaft speed is 2400 rev/min, find the maximum velocity of the valve and the force between cam and valve at its extreme positions.

Ans: 2.51 m/s 648 N (top) 434 N (bottom)