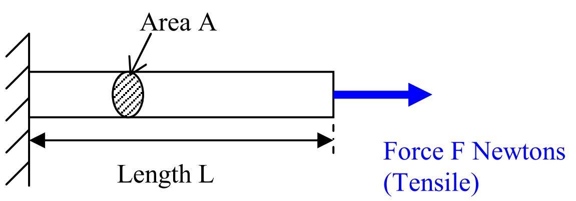
PRINCIPLES OF STRESS AND STRAIN

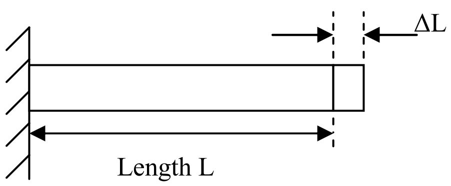
Direct Stress and Strain

When a tensile or compressive load is applied to an engineering component, a direct stress is produced which depends on the magnitude of the load and the cross sectional area of the component.



The stress, σ, is given by $\frac{\text{force}}{\text{area}}$ and the units are N/m^2. Therefore $\sigma = \frac{F}{A}$

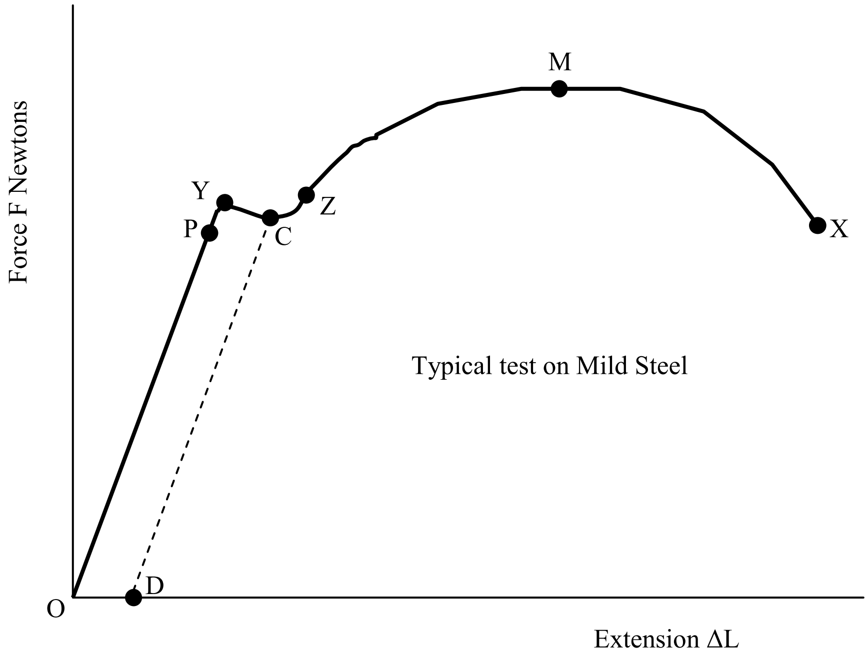
The component will also change its original length L by a very small amount, ΔL or x. This is called strain.



The strain, ε, is given by $\frac{\text{change in length}}{\text{original length}}$ and there are no units. Therefore $\varepsilon = \frac{\Delta L}{L}$

Tensile testing

To determine some material properties, a tensile test to destruction is often carried out. A graph of applied force against extension can be drawn.



The **Elastic Stage** is when the component is unloaded and it returns to its original unstretched length. This is represented by the line OP, and the material is said to obey Hooke's Law.

The point P represents the **Limit of Proportionality**. Beyond P the material no longer obeys Hooke's law.

At Y the material stretches without further increase in load. Y is termed the **Yield Point** and the corresponding stress is the **Yield Stress**.

Beyond Y the material is said to be plastic. If the component is unloaded from any point C, the permanent extension would be OD.

At point Z, further extension requires an increase in load and the material is said to work harden or increase in strength. The process of cold working represents a work hardening or strengthening of this type.

Point M represents the maximum load the component can carry. The test piece begins to neck down or waist, the cross sectional area at the necking point decreases rapidly. At this point the **Ultimate Tensile Stress** can be found and is given by

$$\frac{\text{Maximum Load}}{\text{Original Cross Sectional Area}}$$

The component now increases in length without any additional load being needed. At point X the component fractures. The **Nominal Fracture Stress** or **Breaking Stress** is given by

$$\frac{\text{Load at Fracture}}{\text{Original Cross Sectional Area}}$$

The **Actual Fracture Stress** or **True Stress** is given by

$$\frac{\text{Load at Fracture}}{\text{Final Area at Fracture}}$$

This generally means that

Nominal Fracture Stress < Ultimate Tensile Stress < Actual Fracture Stress

When using materials or components in an engineering situation, a **Factor of Safety** is often used to ensure that the component remains within safe limits when loaded. The Factor of Safety is given by

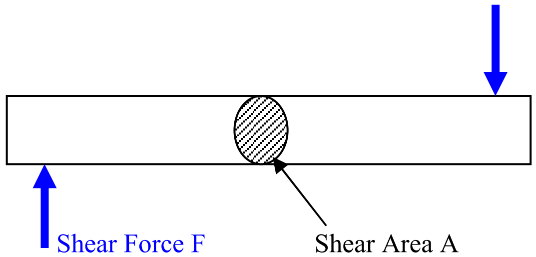
$$\frac{\text{Ultimate Tensile Stress}}{\text{Allowable or Working Stress}}$$

The **Modulus of Elasticity** or **Young's Modulus** (symbol E N/m^2) is the ratio of stress to strain at a point on the initial straight line portion of the load against extension graph. Therefore

$$E = \frac{\sigma}{\varepsilon}$$

Shear Stress

A shear force is one which tries to break the component across its width.



The Shear Stress, τ, is given by $\frac{\text{shear force}}{\text{shear area}}$ and units are N/m^2. Therefore $\tau = \frac{F}{A}$

WORKED EXAMPLES

Example Number 1

A pump rod has a diameter of 50 mm and is 750 mm long. Under working conditions the stress is to be 90 MN/m^2. Calculate

(a) The maximum compressive force to be carried by the rod.
(b) The maximum contraction of the rod when working.

Take E (Modulus of Rigidity or Young's Modulus) for the rod material as 200 GN/m^2.
(a)

$$\text{Compressive Stress} = \frac{\text{compressive force}}{\text{area}} \quad \text{or} \quad \sigma = \frac{F}{A}$$

$$\text{Therefore } F = \sigma \times A$$

$$F = 90 \times 10^6 \times \frac{\pi}{4} \times 0.05^2$$

$$F = \underline{\mathbf{176.7\ kN}} \text{ Answer}$$

$$\text{Young's Modulus} = \frac{\text{stress}}{\text{strain}} \quad \text{or} \quad E = \frac{\sigma}{\varepsilon}$$

$$\text{Therefore } \varepsilon = \frac{\sigma}{E}$$

$$\varepsilon = \frac{90 \times 10^6}{200 \times 10^9}$$

$$\varepsilon = 0.00045$$

(b)

$$\text{Also Strain} = \frac{\text{change in length}}{\text{original length}} \quad \text{or} \quad \varepsilon = \frac{\Delta L}{L}$$

$$\text{Therefore } \Delta L = \varepsilon \times L$$

$$\Delta L = 0.00045 \times 750 \text{ mm}$$

$$\Delta L = \underline{\mathbf{0.3375\ mm}} \text{ Answer}$$

Example Number 2

(a) A steel specimen, 14 mm diameter and gauge length 50 mm, extends 2.5 mm under a tensile load of 1.617 MN. Determine the Modulus of Elasticity for the material.
(b) If the Ultimate Tensile Stress for the material is 400 MN/m^2 and a factor of safety of 2 is required, what would be the maximum allowable extension of a tie bar 3 m long?

(a)

$$\sigma = \frac{F}{A}$$

$$\text{Therefore } \sigma = \frac{1.617 \times 10^6}{\frac{\pi}{4} \times 0.014^2} = 10.5 \text{ GN/m}^2$$

$$\varepsilon = \frac{\Delta L}{L}$$

$$\text{Therefore } \varepsilon = \frac{2.5}{50} = 0.05$$

$$E = \frac{\sigma}{\varepsilon}$$

$$\text{Therefore } E = \frac{10.5 \times 10^9}{0.05} = \underline{\mathbf{210\ GN/m^2}} \text{ Answer}$$

(b)

$$\text{Working Stress} = \frac{\text{Ultimate Tensile Stress}}{\text{Factor of Safety}} = \frac{400}{2} = 200 \text{ MN/m}^2$$

$$\text{Therfore allowable strain } \varepsilon = \frac{\sigma}{E} = \frac{200 \times 10^6}{210 \times 10^9} = 0.0009524$$

$$\text{Therefore allowable extension } \Delta L = \varepsilon \times L = 0.0009524 \times 3000 \text{ mm}$$

$$\Delta L = \underline{\mathbf{2.857\ mm}} \text{ Answer}$$

STUDENT EXAMPLES

1. The tensile load to which a pump rod is subjected is 240 kN. If the tensile stress in the material of the rod is to be 40 MN/m^2 calculate the diameter of the rod.

(87.4 mm)

2. An engine piston rod has to withstand a compressive load of 500 kN. If its diameter is 75 mm what will be the stress?

(113.2 MN/m^2)

3. A hole of diameter 40 mm is to be punched through a plate of thickness 30 mm. If the shearing strength of the metal is 250 MN/m^2 calculate the force required on the punch.

(942.5 kN)

4. A knuckle joint connecting two tie bars carries a load of 100 kN. The pin is 25 mm in diameter and is in double shear. Calculate the shear stress in the pin.

(101.9 MN/m^2)

5. A steel rod diameter 120 mm and 750 mm long has to withstand a tensile force of 680 kN. If the Modulus of Elasticity is 200 GN/m^2 find:-

(a) The stress induced.
(b) The strain.
(c) The bar extension.

(60.13 MN/m^2; 300.7×10^{-6}; 0.2255 mm)

6. A brass tube, Modulus of Elasticity 84 GN/m^2, 280 mm outside diameter and 25 mm wall thickness is 3.5 m long. It is subjected to a stress of 70 MN/m^2. Calculate:-

(a) The working load.
(b) The change in length.

(1402 kN; 3.06 mm)

7. A rivet hole 18 mm diameter is to be punched through a plate 45 mm thick. If the shear strength of the plate material is 280 MN/m^2 calculate:-

(a) The force to be exerted by the punch.
(b) The compressive stress in the punch.

(712 kN; 2800 MN/m^2)

8. A tie bar has a cross sectional area of 400 mm^2 and its extension under load is not to exceed 0.5 mm per metre length. Calculate the maximum load, in kN, which may be applied to the bar. E is 210 GN/m^2.

(42 kN)

9. A tie rod 1 m long and 25 mm diameter is pinned at one end by a bolt which is in single shear. The tension in the rod is 60 kN and its Modulus of Elasticity is 210 GN/m^2. Calculate:-

(a) The stress in the tie rod.
(b) The extension of the rod.
(c) The bolt diameter if the allowable shear stress is to be 60 MN/m^2.

(122.2 MN/m^2; 0.5819 mm; 35.7 mm)

(35.4 mm; 0.3016 mm)

10. The following data was obtained from a tensile test on a mild steel bar of 25 mm diameter and 250 mm gauge:-

Load (kN)	50	100	150	160	170	180	190	200	205	210	215
Ext (mm)	0.12	0.24	0.37	0.39	4.3	4.6	6.6	8.1	9.2	10	11

Load (kN)	220	225	230	235	240	245	250	255	251	231	217
Ext (mm)	12.5	13.5	15	17.5	20	22.5	27.5	37.1	54	58.5	59.6

Plot separate load against extension graphs for the elastic range and full range and hence determine:-
(a) The yield stress.
(b) U.T.S.
(c) Young's Modulus of Elasticity.

(339 MN/m^2; 520 MN/m^2; 207 GN/m^2)

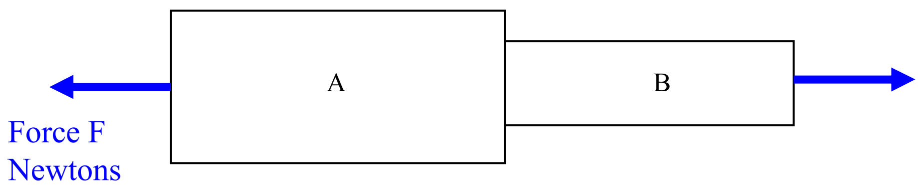
11. An engine piston rod has to withstand a compressive load of 500 kN. If its diameter is 75 mm and the ultimate tensile stress for the metal is 400 MN/m^2, what will be the factor of safety?

(3.53)

COMPOUND BARS

Compound Bars in Series

Sometimes components are joined together end to end. This is termed a compound bar where the components are in series.



The two components A and B may be different materials (e.g. one steel and one copper), they may be different lengths, they may be different cross sectional areas.

The force F Newtons will be the same force applied to each component. Therefore the stresses and strains in each component will be

$$\sigma_A = \frac{F}{A_A} \text{ and } \sigma_B = \frac{F}{A_B}$$

$$\varepsilon_A = \frac{\Delta L_A}{L_A} \text{ and } \varepsilon_B = \frac{\Delta L_B}{L_B}$$

The total change in length of the compound bar will be

$$\Delta L_{TOTAL} = \Delta L_A + \Delta L_B$$

If the materials are different then

$$E_A = \frac{\sigma_A}{\varepsilon_A} \text{ and } E_B = \frac{\sigma_B}{\varepsilon_B}$$

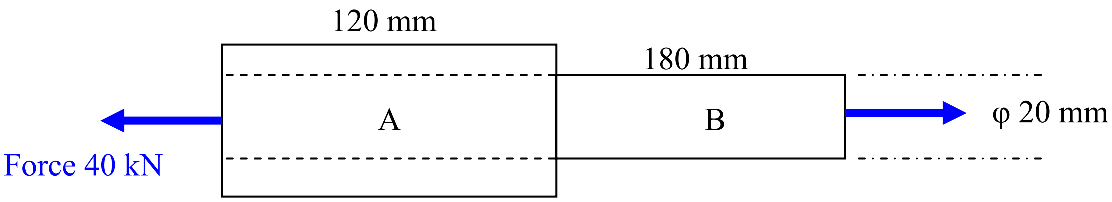
However, if the materials are the same then

$$E = \frac{\sigma_A}{\varepsilon_A} = \frac{\sigma_B}{\varepsilon_B}$$

WORKED EXAMPLE

(a) A solid, cylindrical steel bar 20 mm diameter and 180 mm long is welded to the end of a hollow steel tube of 20 mm inside diameter and 120 mm long. A load of 40 kN is applied to the compound bar. If the stress is constant along the total length of the compound bar, find the outside diameter of the steel tube.

A diagram is always useful:



The question says that $\sigma_A = \sigma_B$. Therefore $\frac{F}{A_A} = \frac{F}{A_B}$ and since the force applied to each component is the same then the areas must be equal i.e. $A_A = A_B$. If the outside diameter of component A is said to be 'D' then

$$\frac{\pi}{4} \times 20^2 = \frac{\pi}{4} \times (D^2 - 20^2)$$

The $\frac{\pi}{4}$ term will cancel to give $20^2 = D^2 - 20^2$ and solving will give

D = **<u>28.28 mm</u>** Answer.

(b) What is the stress in each component?

Again, the question says that the stress in each component is the same, and $\sigma = \frac{F}{A}$. Therefore, use either the area of the solid component or the hollow component (these two areas are the same, see part (a) of the question).

$$\sigma = \frac{F}{A_B} = \frac{40 \times 10^3}{\frac{\pi}{4} \times 0.02^2} = \underline{127.3 \ \text{MN/m}^2} \text{ Answer}$$

(Check this answer using the area of component A, it should be the same value)

(c) If Young's Modulus for steel is 207 GN/m^2, what is the strain in each component?

The strain is $\varepsilon = \frac{\sigma}{E} = \frac{127.3 \times 10^6}{207 \times 10^9} = \underline{\mathbf{0.0006155}}$ Answer

This will be the same value for each component since they are made from the same material and their stresses are the same. However, will their change in lengths be the same? If not, why not?

(d) What is the total change in length?

Even though the strains are the same, the lengths of each component is different. Therefore, each will have a different extension.

$$\Delta L_A = \varepsilon_A \times L_A = 0.0006155 \times 180 = 0.1108\,\text{mm}$$
$$\Delta L_B = \varepsilon_B \times L_B = 0.0006155 \times 120 = 0.07386\,\text{mm}$$

$$\Delta L_{TOTAL} = \Delta L_A + \Delta L_B = 0.1108 + 0.07386 = \underline{\mathbf{0.1847\ \ mm}}\ \text{Answer}$$

Typical examination question

A steel rod of diameter 40mm and length 100mm is firmly attached at one end to a copper rod of length 150mm. An axial tensile force of 60kN is applied to the 250 mm long composite rod, and both materials extend by the same amount.

Calculate EACH of the following:

a)	The diameter of the copper rod	(6)
b)	The total extension	(3)
c)	The stress in each rod	(4)
d)	The strain energy stored in the steel rod	(3)

The Modulus of Elasticity for the steel = 210 GN/m^2
The Modulus of Elasticity for the copper = 90 GN/m^2

This question deals with a compound bar *in series* that is loaded with an axial load.

Draw the rod showing the details given in the question stem.

When the common load of 60 kN is applied it will extend both the steel and copper bars. The comment that these extensions are the same is the main clue to the question.

From $E = \dfrac{\sigma}{\varepsilon}$, then $x = \dfrac{Fl}{AE}$,

thus for the steel bar then the extension x_s = $\dfrac{60000 \times 0.1}{210 \times 10^9 \times \dfrac{\pi 0.04^2}{4}} = 0.0227$ mm

This extension is the same for the copper bar, so the area of the copper bar A_b is

$\dfrac{60000 \times 0.015}{90 \times 10^9 \times 0.0227 \times 10^3} = 4.4 \times 10^{-3}$ m^2.

Thus the diameter of the copper bar is 74.83 mm

The total extension of the compound bar will be the addition of the extension of each bar, or double the extension of each component of the bar.
So x_{total} = 2 x 0.0227 = 0.0454 mm

The stress in each rod is simply $\dfrac{F}{A}$ so σ_{steel} = $\dfrac{60000}{\dfrac{\pi 0.04^2}{4}} = 47.75$ MN/m^2.

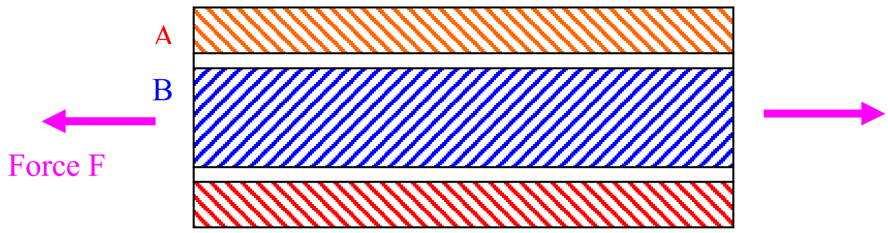
so $\sigma_{copper} = \dfrac{60000}{\dfrac{\pi 0.07483^2}{4}} = 136.43$ kN/m^2

The final part is to find the strain energy (SE)

So SE = $\dfrac{Fx}{2} = \dfrac{60000 \times 0.0227 \times 10^{-3}}{2} = 0.68$ J

Compound Bars in Parallel

If the two components are parallel with each other rather than being end to end, the problem becomes slightly more complicated. Examples of this are circular shafts inside hollow bushes, holding down bolts in cover plates etc.



The force F is now shared between the two components. The problem is how much of the force is taken by component A and how much is taken by component B. Whatever the value, both added together must equal the total force F.

$$F = F_A + F_B$$

and $\sigma_A = \dfrac{F_A}{A_A}$ or $F_A = \sigma_A \times A_A$. Similarly $F_B = \sigma_B \times A_B$.

$$\text{Therefore } F = \sigma_A \times A_A + \sigma_B \times A_B \qquad \text{Equation (1)}$$

In most problems, both components are assumed to have the same original length and they undergo the same change in length (the strains are equal).

Therefore

$$\varepsilon_A = \varepsilon_B$$

$$\frac{\Delta L_A}{L_A} = \frac{\Delta L_B}{L_B}$$

$$\text{or } \frac{\sigma_A}{E_A} = \frac{\sigma_B}{E_B} \qquad \text{Equation (2)}$$

There are two equations, where the two unknowns are usually the stresses.

WORKED EXAMPLE

A steel bar of diameter 50 mm and 80 mm long is placed inside a brass tube of inside diameter 60 mm and outside diameter 75 mm. The tube is also 80 mm long. A total load of 15 kN is applied to the compound bar. Take Young's Modulus for each material as $E_S = 207\ GN/m^2$ and $E_B = 83\ GN/m^2$.

Calculate:
(a) The stress in each material.
(b) The force in each material.
(c) The strain in each material.
(d) The change in length of each component.

(a) The total force of 15 kN is shared between the two components.

$$F_{TOTAL} = F_S + F_B$$
$$F_{TOTAL} = \sigma_S \times A_S + \sigma_B \times A_B$$
$$15\times10^3 = \sigma_S \times \frac{\pi}{4}\times 0.05^2 + \sigma_B \times \frac{\pi}{4}\times(0.075^2 - 0.06^2)$$
$$15\times10^3 = \sigma_S \times 0.001963 + \sigma_B \times 0.00159 \qquad \text{Equation (1)}$$

Due to the load being applied, the strain of each component must be the same.

$$\varepsilon_S = \varepsilon_B$$
$$\text{Therefore} \quad \frac{\sigma_S}{E_S} = \frac{\sigma_B}{E_B}$$
$$\sigma_S = \frac{E_S}{E_B}\times\sigma_B$$
$$\sigma_S = \frac{207\times10^9}{83\times10^9}\times\sigma_B$$
$$\sigma_S = 2.494\times\sigma_B \qquad \text{Equation (2)}$$

Substitute equation (2) into equation (1)

$$15\times10^{£} = 2.494\times\sigma_B\times0.001963 + \sigma_B\times0.00159$$
$$15\times10^3 = 0.006486\times\sigma_B$$
$$\sigma_B = \underline{\mathbf{2.312\ MN/m^2}} \text{ Answer}$$

$$\sigma_S = 2.494\times\sigma_B = 2.494\times2.312 = \underline{\mathbf{5.766\ MN/m^2}} \text{ Answer}$$

(b) The force is given by $F = \sigma \times A$

Therefore for the steel

$$F_S = 5.766 \times 10^6 \times 0.001963$$
$$F_S = \underline{\mathbf{11.32\,kN}} \quad \text{Answer}$$

and for the brass

$$F_B = 2.312 \times 10^6 \times 0.00159$$
$$F_B = \underline{\mathbf{3.69\,KN}} \quad \text{Answer}$$

NOTE! Both of these forces add up to 15 kN, the total force.

(c) The strain in each component is the same. Therefore

$$\varepsilon = \frac{\sigma_S}{E_S} = \frac{\sigma_B}{E_B}$$
$$\varepsilon = \frac{5.766 \times 10^6}{207 \times 10^9}$$
$$\varepsilon = \underline{\mathbf{0.00002786}} \quad \text{Answer}$$

Check this answer using the values for the brass.

(d) The change in length will also be the same for each component since their original lengths (80 mm) are the same.

$$\Delta L = \varepsilon \times L$$
$$\Delta L = 0.00002786 \times 80$$
$$\Delta L = \underline{\mathbf{0.002229\,mm}} \quad \text{Answer}$$

STUDENT EXAMPLES

1. A steel tube with an outside diameter of 80 mm and a bore of 50 mm is bushed with a bronze tube of 50 mm outside diameter and 30 mm bore. Treating the assembly as a compound tube, find the stress in each material if it is subjected to an axial compressive stress of 50 kN. If the length of the tube is 200 mm, determine also the reduction in length.
$E_{steel} = 200$ GN/m^2 and $E_{brass} = 90$ GN/m^2.

($\sigma_{steel} = 13.75$ MN/m^2; $\sigma_{brass} = 6.18$ MN/m^2; 0.01375 mm)

2. A beam of mass 1220 kg is suspended in a horizontal position by three vertical wires fixed to a rigid support. The outer wires are brass and the central wire is steel, each having a cross sectional area of 160 mm^2 and a length of 4 m. Determine the stress in each material and the extension of the wires.
$E_{steel} = 200$ GN/m^2 and $E_{brass} = 80$ GN/m^2.

($\sigma_{steel} = 16.67$ MN/m^2; $\sigma_{brass} = 41.67$ MN/m^2; 0.833 mm)

3. Two vertical wires, one of steel and one of copper, are fastened 1 m apart at their upper ends. They are each of the same length and the diameter of the copper wire is twice the diameter of the steel wire. If a bar supporting a load is attached to their lower ends, determine where the load should be hung from the bar so that the bar remains horizontal.
$E_{steel} = 220$ GN/m^2 and $E_{copper} = 110$ GN/m^2.

(⅓ of a meter from the copper wire)

4. A uniform beam, 4 m long, is simply supported at its ends by two struts, one of copper and one of steel. The diameter of the copper strut is 40 mm and the steel strut 30 mm. If a vertical force of 20 kN is applied to the beam and the beam remains horizontal, how far from the steel strut is this force applied.
$E_{steel} = 200$ GN/m^2 and $E_{copper} = 100$ GN/m^2.

(1,88 m from the steel strut)

5. A steel rod of 25 mm diameter and 200 mm long stands co-axially with a brass tube of 40 mm outside diameter and 30 mm bore which is 180 mm long. The base support is rigid and recessed so that the upper ends of the rod and tube are in line. A rigid upper plate is now placed over the top of the assembly and a compressive force of 90 kN is applied to the plate. Determine the stresses induced in the two components and the resulting reduction in length.
$E_{steel} = 200$ GN/m^2 and $E_{brass} = 80$ GN/m^2.

($\sigma_{steel} = 123$ MN/m^2; $\sigma_{brass} = 54.4$ MN/m^2; 0.12 mm)