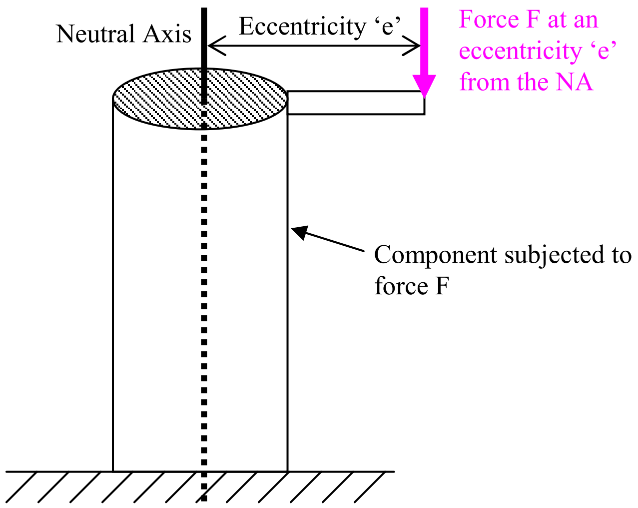
COMBINED BENDING AND DIRECT STRESSES

Look at the diagram below.

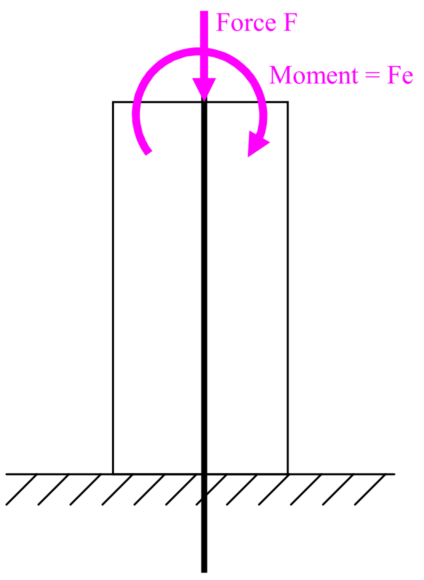


The component has a force F Newtons applied to it, which, in this drawing, is offset from the Neutral Axis by an eccentricity 'e' metres. This force causes both a direct compressive stress in the component and also a bending stress due to the moment 'Fe' Nm.

The bending stress will be compressive on one side of the NA (in this example the RHS) and tensile on the other side of the NA (in this example the LHS).

The direct and bending stresses can be calculated separately and then added together (remembering that tensile is positive and compressive is negative) to give the resultant stress. This is called the **PRINCIPLE OF SUPERPOSITION** where a complex problem is broken down into simpler elements.

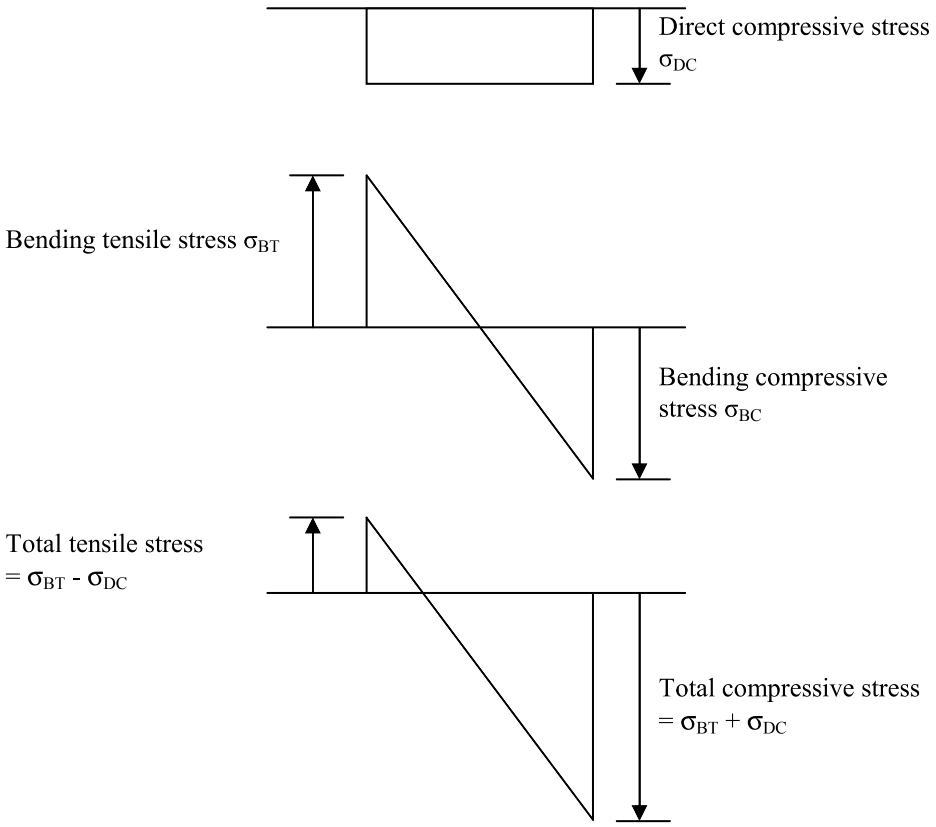
The above diagram can be replaced by:



The direct stress will be given by $\sigma = \dfrac{F}{A}$ and will be compressive.

The bending stress will be given by $\sigma = \dfrac{My}{I} = \dfrac{Fey}{I}$ and will be both tensile and compressive.

These stresses can be represented graphically:



WORKED EXAMPLE

A short steel tube of outside diameter 128 mm and inside diameter 80 mm has an offset compressive force applied which is parallel to the longitudinal neutral axis of the tube. The maximum stresses produced are found to be 3 MN/m^2 tensile and 15 MN/m^2 compressive.
(a) Calculate the magnitude of the applied compressive force.
(b) Calculate the offset of the applied compressive force from the neutral axis.
(c) Calculate where the stress is zero from the neutral axis.

(a) Adding the direct and bending compressive stresses must give 15 MN/m^2. Taking tensile as positive and compressive as negative:

$$-\sigma_{DC}-\sigma_{BC}=-15 \text{ Equation (1)}$$

Adding the direct compressive stress and bending tensile stress gives a total tensile stress of 3 MN/m^2:

$$-\sigma_{DC}+\sigma_{BT}=3 \text{ Equation (2)}$$

At first there appears to be three unknowns, but only two equations. However, because the tube is circular and the NA is in the centre of it, then the magnitudes of the two bending stresses must be the same.

$$\text{i.e. } \sigma_{BC}=\sigma_{BT}=\sigma_B \text{ Equation (3)}$$

Therefore, adding (1) + (2):

$$-2\sigma_{DC}=-12$$
$$\sigma_{DC}=6 \text{ MN/m}^2 \text{ compressive}$$

and

$$\sigma_{DC}=\frac{F}{A}$$
$$F=\sigma_{DC}\times A$$
$$F=6\times10^6\times\frac{\pi}{4}(0.128^2-0.08^2)$$

F = **47.05 kN** Answer

(b) The bending stress is given from either equation (1) or (2). Using equation (1) gives:

$$-\sigma_{DC}-\sigma_{BC}=-15$$
$$-6-\sigma_{BC}=-15$$
$$\sigma_{BC}=15-6$$
$$\sigma_{BC}=9 \text{ MN/m}^2$$

The bending stress is given by $\frac{My}{I} = \frac{Fey}{I}$. Therefore

$$\sigma_B = \frac{Fey}{I}$$

$$e = \frac{\sigma_B I}{Fy}$$

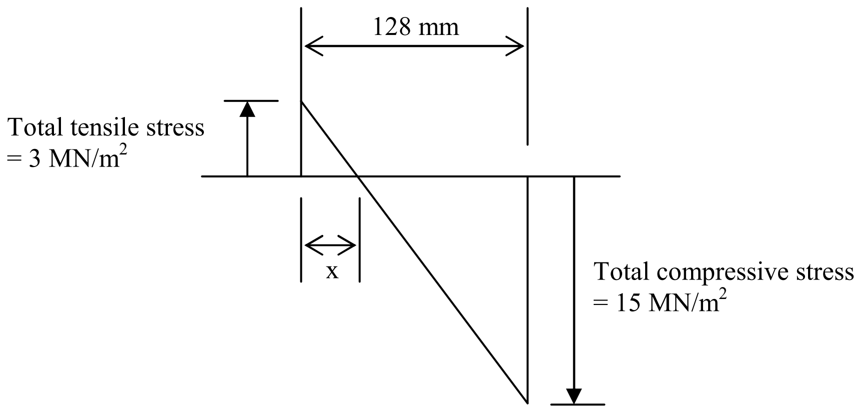
$$e = \frac{9\times10^6 \times \frac{\pi}{64}(0.128^4 - 0.08^4)}{47.05\times10^3 \times \frac{0.128}{2}}$$

$$e = \frac{100.5}{3009}$$

$$e = 0.0334 \text{ m}$$

$e =$ **33.4 mm from the neutral axis** Answer

(c) A diagram of stress distribution across the diameter of the tube will be:



From similar triangles:

$$\frac{3}{x} = \frac{15}{128 - x}$$

$$3(128 - x) = 15x$$

$$384 - 3x = 15x$$

$$18x = 384$$

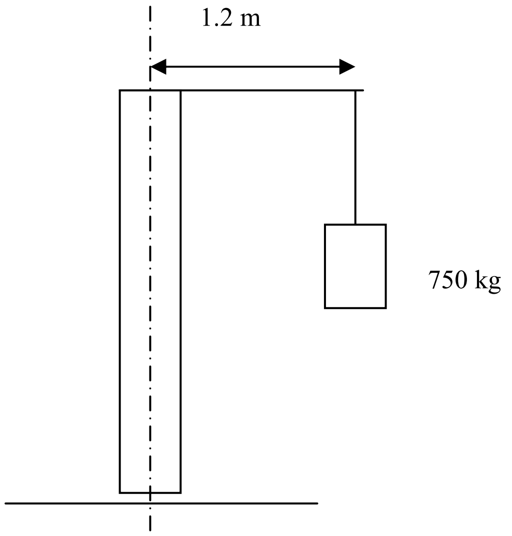
$$x = \frac{384}{18}$$

$x =$ **21.33 mm** Answer

Typical examination question

A lifting arm as shown, is to be designed to lift 750kg at an extension of 1.2 metre, measured from the centre of the tube centre. Tubing of 150mm bore is to be used as the vertical tube, but the combined bending and direct stress is limited to 380 MN/m^2.

Using a safety factor of 4, determine the thinnest wall thickness of the vertical tube that will satisfy the stress limitation from either 4.0, 5.0, or 7.0mm tube wall (16)



This question considers the combined stress imposed on the tube. This will be a combination of

1. direct compressive stress, and
2. bending stress, which will produce both additional tensile and compressive stress on the tube.

The direct compressive stress = $\dfrac{F}{A} = \dfrac{750g}{A}$

The bending stress will be $\sigma = \dfrac{My}{I} = \dfrac{(750g \times 1.2)0.075}{I}$, where M = 750g x 1.2

The total stress allowed is $\dfrac{380}{4} = 95$ MN/m^2, and in this case the maximum stress will be compressive in nature, as the compressive stress is added to the compressive stress produced by the bending.

Calculating the value of area A, and second moment of area I for each possible selection of tube.

Thickness	Inside diameter	Area	Second moment
4 mm	142 mm	1.83 x 10^{-3} m^2	4.89 x 10^{-6} m^4
5 mm	140 mm	2.28 x 10^{-3} m^2	6.0 x 10^{-6} m^4
7 mm	136 mm	3.14 x 10^{-3} m^2	8.06 x 10^{-6} m^4

$$A = \frac{\pi(D^2 - d^2)}{4}$$

$$I = \frac{\pi(D^4 - d^4)}{64}$$

Calculating for the 5 mm thickness (this will allow me to calculate either side, so only doing two rather three solutions.

The direct compressive stress = $\dfrac{F}{A} = \dfrac{750g}{A} = \dfrac{750g}{2.28\times10^{-3}} = 3.23$ MN/m^2

The bending stress will be $\sigma = \dfrac{My}{I} = \dfrac{(750g\times1.2)0.075}{6\times10^{-6}} = 110.36$ MN/m^2

The total compressive stress will be 3.23 + 110.36 which exceeds the 95 limit.

Calculating for the 7mm thickness

The direct compressive stress = $\dfrac{F}{A} = \dfrac{750g}{A} = \dfrac{750g}{3.14\times10^{-3}} = 2.34$ MN/m^2

The bending stress will be $\sigma = \dfrac{My}{I} = \dfrac{(750g\times1.2)0.075}{8.06\times10^{-6}} = 82.15$ MN/m^2

Hence the maximum compressive stress in this situation will be 2.34 + 82.15 = 84.49 MN/m^2 which is below the allowable maximum.

STUDENT EXAMPLES

1. A column of 150 mm outside diameter and 50 mm inside diameter is subjected to a non-axial compressive force F. If the maximum and minimum compressive stresses set up in the column as a result of this force are 40 MN/m^2 and 15 MN/m^2, find the force F and its eccentricity.

(431 kN; 9.46 mm)

2. A boiler stay of 50 mm nominal diameter is designed to withstand a maximum stress which, due to the threads not being concentric with the axis of the stay, is 12% above the normal direct stress. Determine the maximum allowable eccentricity of the screw thread circle with the stay axis.

(0.75 mm)

3. A short hollow cylindrical cast iron pillar is 250 mm outside diameter and 25 mm thick. It is subjected to a compressive force of 400 kN, the line of action of which is 20 mm from the centre of the pillar. Find the maximum and minimum stresses induced in the pillar and determine also the maximum permissible eccentricity of this force so that one side is just about to be in tension.

(31.4 MN/m^2 compressive; 13.8 MN/m^2 compressive; 51 mm)

4. A short hollow cylindrical column, of outside diameter twice the inside diameter, is subjected to a non-axial compressive force F. Show that there will be no tensile stress present if the eccentricity of this force does not exceed $\frac{5}{16}$ of the inside diameter. A hollow column of 300 mm outside diameter and 150 mm inside diameter is subjected to a compressive force of 250 kN, offset from the axis of the column by 80 mm. Find the maximum stresses induced and sketch the stress distribution diagram.

(12.52 MN/m^2 compressive; 3.52 MN/m^2 tensile)

5. A short cast iron column is of retangular section. Its external dimensions are 200 mm × 150 mm, it is 30 mm thick and is subjected to a vertical compressive force which acts at 60 mm from the axis of the column and on the centre line parallel to the shorter side. If the maximum permissible stresses are 62.8 MN/m^2 compressive and 15.2 MN/m^2 tensile, calculate the greatest permissible force.

(413 kN)

6. From strain measurements made on the surface of ashort pillar, consisting of a tube 150 mm outside diameter and 12.5 mm thick, it is found that the maximum and minimum stresses induced, by the application of acompressive load, are 100 MN/m^2 and 50 MN/m^2 respectively, both compressive. Determine the magnitude of this load and its deviation from the axis of the pillar.

(404 kN; 10.6 mm)

7. A short column is of hollow section 200 mm outside diameter and uniform thickness 40 mm. A vertical compressive force acts at 70 mm from the axis of the column. If the maximum permitted streses are 45 MN/m^2 compressive and 12 MN/m^2 tensile, find the greatest allowable force. Assuming this force is acting, sketch te stress distribution diagram across the section and from this determine the position of zero stress relative to the surface in tension.

(296 kN; 42.2 mm from surface in tension)

8. A hollow cylindrical column 600 mm long, 100 mm outside diameter and 80 mm bore, is subjected to a compressive force of 10 kN at the top which is inclined at 30^0 to the axis of the column. Calculate the maximum compressive and tensile stresses set up in the base of the column.

(55.06 MN/m^2 compressive; 48.94 MN/m^2 tensile)

9. A temporary crane hook is made from steel bar of 60 mm × 30 mm section and the hook, when formed, has an inside radius of 50 mm. Determine the maximum tensile and compressive stresses induced in the material when a mass of 2 tonne is suspended from the hook.

(98.15 MN/m^2 tensile; 76.35 MN/m^2 compressive)

10. The main frame of a punching machine is of I section with a cross sectional area of 225 cm^2 and a second moment of area 12.5 × 10^4 cm^4. The outer face of the flange nearest to the punch is 300 mm from the axis of the section and the set from the centre line of the punch to the axis of the section is 1 m. If the maximum permitted stress in the frame is limited to 32 MN/m^2, find the maximum force exerted at the punch.

(112.5 kN)

11. A crane hook is of elliptical cross section with major and minor axes of 70 mm and 35 mm respectively. The inside radius of the hook is 115 mm and it supports a load of 6 kN. The second moment of area for the section is given by $\frac{\pi}{64}a^3b$ where a is the major axis length and b is the minor axis length. The area of an ellipse is πab. Find the maximum tensile and compressive stresses induced.

(56.52 MN/m^2 tensile; 50 28 MN/m^2 compressive)

12. The column of a portable drill is of solid circular section 50 mm diameter and the centre line of the drill is 200 mm from the axis of the column. If the maximum permissible stress in the column is not to exceed 60 MN/m^2 find the maximum force allowed when drilling.

(3.58 kN)

EULER'S THEORY FOR LONG SLENDER STRUTS

In the section on Combined Bending and Direct Stresses it was shown that an offset load from the neutral axis produces both direct and bending stresses. If the load were placed directly on the neutral axis then only a direct compressive (or tensile) stress would result.

However, this only applies if the component is considered to be very short in comparison to its cross section dimensions. The longer the length becomes, then the more likely the component is to fail by **BUCKLING**.

A strut is a member subjected to a direct compressive stress. The load carrying capacity of relatively short struts with large cross section area is limited by the crushing strength of the material. Long and slender struts, however, can become unstable and tend to buckle.

A small transverse load applied to the mid-point of a slender strut will produce a lateral deflection, which disappears when the transverse load is removed. As the compressive load is increased a point is reached at which the lateral deflection does not disappear. At this point the strut is in a state of unstable equilibrium and the slightest lateral disturbance will cause it to buckle. Such a strut has clearly reached the limit of its load carrying capacity, and the load is said to have reached its critical value.

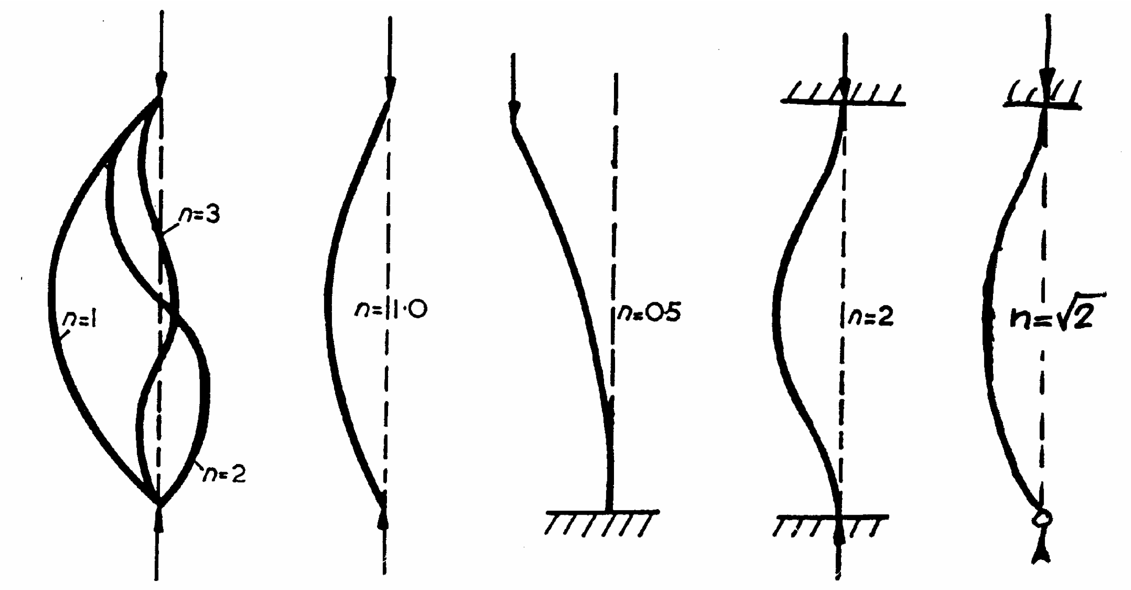
The critical load for a strut may be found using **EULER'S THEORY**, which is based on the assumptions below:

1. The material is homogenous.
2. The load is applied axially at the centroid of the section.
3. The cross section is uniform.
4. The strut is initially straight.
5. The direct stresses due to the compressive load are negligible compared with the bending stresses induced by buckling.

It can be shown that Euler's Critical load, F_E Newtons, is given by:

$$\boxed{F_E = \frac{n^2\pi^2 EI}{L^2}}$$

F_E is the critical Euler load to cause buckling.
E is Young's Modulus.
I is the MINIMUM second moment of area of the cross section of the strut.
L is the length of the strut.
n depends on the manner of buckling. This is a number which is determined from the way in which the strut is supported. There are numerous ways in which the strut can be supported, but the main ones are:



(a) (b) (c) (d) (e)

Diagram (a) shows a strut pin jointed at each end. There are several modes of failure as illustrated, but it is the smallest value of load, and therefore n, which will cause failure. In this case the failure will be when n = 1 as in diagram (b).

Diagram (c) shows a strut built in at one end and free to move at the other.

Diagram (d) shows a strut built in at both ends.

Diagram (e) shows a strut built in at one end and allowed to move only vertically at the other.

Each of the above diagrams shows that the strut will deflect into a complete or part sine wave. The value of n is the **NUMBER OF COMPLETE HALF SINE WAVES**. The larger the value of n (and hence the larger the number of half sine waves) the larger the load needed to cause buckling. If the strut can be supported along its length as often as possible so that no lateral movement is possible, the value of n increases and the strut becomes more stable.

Effective Length of a strut

If a strut is made to deflect into two half sine waves, then it could be considered as two separate struts which are in series, each being half the length of the original strut. The critical load will then be greater.

In general, if a strut is made to deflect into n half sine waves, then the strut becomes 'n struts' each of length $\frac{L}{n}$ where L is the length of the original strut. Therefore

$$\text{Effective Length } l = \frac{L}{n}$$

The Euler equation then becomes

$$F_E = \frac{\pi^2 EI}{l^2}$$

Slenderness Ratio

Since $I = Ak^2$ where A is the cross sectional area of the strut and k is the MINIMUM radius of gyration, then Euler's equation can be written as

$$F_E = \frac{n^2\pi^2 EAk^2}{L^2} = \frac{\pi^2 EAk^2}{l^2}$$

$$F_E = \frac{\pi^2 EA}{\left(\frac{l}{k}\right)^2}$$

This shows that for a given cross sectional area A, the critical load F_E is inversely proportional to the square of the ratio $\frac{l}{k}$. This determines when instability will start and is called the Slenderness Ratio. For most engineering materials and applications, Euler's equation can only be used when the slenderness ratio is greater than **120**. At values less than this the strut will be stable and will only suffer direct compressive stress.

WORKED EXAMPLE

A strut is 2 m long and has a rectangular cross section 30 mm × 20 mm. It is pin jointed at each end and is constrained to move axially in guides. E for the material is 200 GN/m^2.

(a) What is the value of n to be used in Euler's equation?

This strut is the same type as in diagram (b) in the notes.

Therefore the value of n is 1. Answer

(b) What is the effective length of the strut?

The effective length is $l = \frac{L}{n} = \frac{2}{1} = \underline{\textbf{2 meters}}$ Answer

(c) What is the cross sectional area of the strut?

Area = 30 × 20 = $\underline{\textbf{600 mm}^2}$ Answer

(d) What is the minimum second moment of area I?

The value of I, for a rectangular cross section, is given by $I = \frac{bd^3}{12}$. Therefore the minimum will be when b = 30 mm and d = 20 mm.

$$I = \frac{bd^3}{12} = \frac{30 \times 20^3}{12} = \underline{\textbf{20000 mm}^4} \text{ Answer}$$

(Note! The maximum value of I would be when b = 20 mm and d = 30 mm giving I as 45000 mm^4)

(e) What is the minimum radius of gyration, k?

$$k^2 = \frac{I}{A} = \frac{20000}{600} = 33.33 \text{ mm}$$

$$k = \sqrt{33.33} = \underline{\textbf{5.774 mm}} \text{ Answer}$$

(f) What is the slenderness ratio for the strut?

$$\text{Slenderness ratio} = \frac{l}{k} = \frac{2000\text{ mm}}{5.774\text{ mm}} = \underline{\mathbf{346}}\text{ Answer}$$

This means that since the value is greater than 120, Euler's equation can be used to determine the critical load for buckling.

(g) Determine Euler's Critical Load for this strut.

$$F_E = \frac{\pi^2 EA}{\left(\frac{l}{k}\right)^2}$$

$$F_E = \frac{\pi^2 \times 200 \times 10^9 \times 600 \times 10^{-6}}{346^2}$$

$$F_E = \frac{1184 \times 10^6}{0.1197 \times 10^6}$$

$$F_E = 9891\text{N} = \underline{\mathbf{9.891\ kN}}\text{ Answer}$$

This means that if the load is less than 9.891 kN, the strut will be stable and will only suffer a direct compressive stress. If the load is greater than 9.891 kN, the strut will be unstable and will fail due to buckling.

<u>Typical examination question</u>

A box section strut is loaded axially by a 6kN load. The strut outside dimensions are 100 x 70mm with a 8mm wall thickness.

Calculate EACH of the following
a) The compression per metre length (6)
b) The maximum compression of the strut before buckling occurs (10)

Take Euler's crippling load P_c for this loaded condition as

$$P_c = \frac{\pi^2 EI}{(0.7L)^2}$$

The Modulus of Elasticity of the strut material is 170 GN/m^2

The compression per metre length can be found from the relationship on page two of Section a) of the Strength of Material section.

From $E = \frac{\sigma}{\varepsilon} = \frac{Fl}{Ax}$ then the compression x is $\frac{Fl}{AE}$

The area A is (0.1 x 0.07) – (0.084 x 0.054) = 0.002464 m^2. Note the thickness will make the inside dimensions be 84 by 54 mm. It is a common mistake for students to state the inside dimensions are 92 by 62mm, as only one thickness is used rather than two. Be careful.

Thus the compression x = $\frac{6000 \times 1}{170 \times 10^9 \times 0.002464} = 0.0143$ mm per m of beam

The second moment of area I for a rectangular beam can have two values. However it is the lowest value that we are interested in, as this will give the minimum force.

I for this section is $\frac{BD^3 - bd^3}{12} = \frac{(0.1 \times 0.07^3) - (0.084 \times 0.054^3)}{12} = 1.756 \times 10^{-6}$ m^4

Try to calculate the maximum I value, you should get 3.166 x 10^{-6} m^4

From Euler's relationship $P_c = \frac{\pi^2 EI}{(0.7L)^2}$ the length of the beam that will buckle on a length L when subjected to a 5 kN load is

$$L = \frac{\pi}{0.7}\sqrt{\frac{EI}{P_c}} = \frac{\pi}{0.7}\sqrt{\frac{170 \times 10^9 \times 1.756 \times 10^{-6}}{6000}} = 31.66 \text{ m}$$

Hence the maximum compression will be the compression per m length x the length of the beam or 31.66 x 0.0143 = 0.453 mm

STUDENT EXAMPLES

1. A vertical strut is 16 m long. Its cross section is a symmetrical I section where the thickness of material is 10 mm. The flanges are 250 mm long and the web is 300 mm long. The strut is built in at both ends and E for the material is 200 GN/m^2.
(a) Calculate the minimum value of I.
(b) Calculate the slenderness ratio.
(c) Calculate the critical load.

(a) 26.07×10^{-6} m^4 (b) 140 (c) 805.6 kN

2. A strut consists of a straight metal bar 1 m long and of rectangular cross section 12 mm × 5 mm. It is pin jointed at each end and E for the material is 70 GN/m^2.
(a) Calculate the minimum value of I.
(b) Calculate the slenderness ratio.
(c) Calculate the critical load.

(a) 0.125×10^{-9} m^4 (b) 693 (c) 86.36 N

3. An alloy tube of length 3.2 m has an external diameter of 18 mm and an internal diameter of 13 mm. When subjected to an axial tensile force of 4.5 kN, the extension of the tube was 1.1 mm.
(a) Calculate the modulus of elasticity for the tube material.
(b) The tube is to be used as a vertical strut to carry an axial compressive load.
(i) If the ends are pin jointed, calculate Euler's critical load.
(ii) If one end is built in and the other end free, calculate Euler's critical load.

(a) 107.5 GN/m^2 (b)(i) 388.8 N (ii) 777.6 N