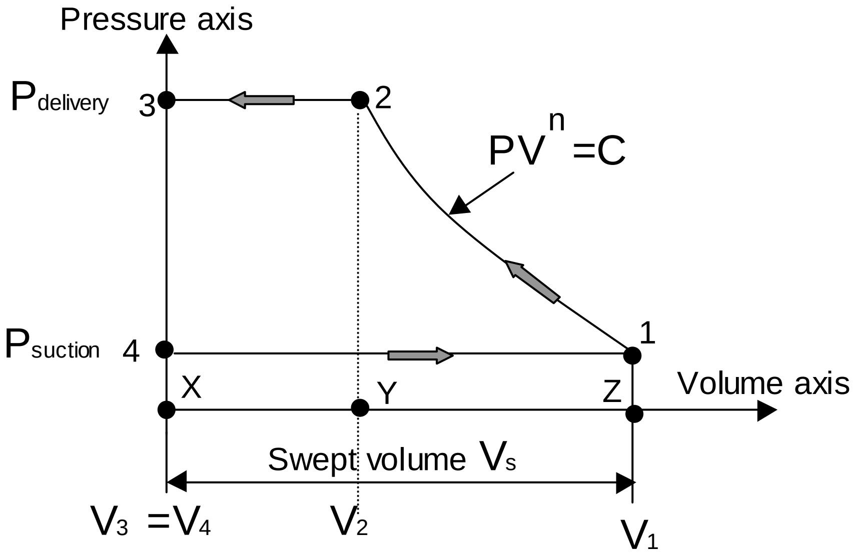
Reciprocating Compressors

The function of a compressor is to take a quantity of gas such as air and deliver it at a required pressure.
Reciprocating and rotary positive displacement machines are used for a variety of purposes, however in general reciprocating machines having the characteristics of low mass flow rates and high pressure ratios, the rotary machines tend to have a high mass flow rates and low-pressure ratios.
Both types exist in various forms each having its, own characteristic and may be single or multistage, and have either air or water, cooling.
The reciprocating machine is pulsating in action, which limits the rate at which fluid can be delivered where as the rotary machine has a continuous action.

These notes will refer only to reciprocating machines in either single or multi-stage arrangements.
It will also be assumed that you are familiar with their construction and operation.
All The diagrams will be for an idealised cycle in that there will be definite corner points on the diagrams and all the lines will be straight.
An actual indicator diagram is similar to the ideal except for the induction and delivery processes, which are modified by valve action. This is shown as waviness in the induction and delivery lines due to valve bounce. Automatic valves are less definite in action than cam-operated valves, they also give more throttling of the gas.
The fact that the gas will be heated by the cylinder walls, and there will be a reduction in pressure due to the pressure drop required to induce the gas into the cylinder against the resistance to flow has been ignored and we will assume that the inlet conditions during the induction stroke are constant.

Single stage compressor without clearance

In this case it is assumed the piston touches the cylinder head and therefore no air is trapped in the cylinder at top dead centre.
As the piston moves to bottom dead centre air is drawn in until the piston reaches bottom dead centre and stops.
This is the swept or stroke volume and with no clearance, is the volume induced.
As the piston moves towards top dead centre the air is compressed until the discharge pressure is reached at which point delivery of the air at constant pressure begins.
The piston moves away from top dead centre and because there is no clearance volume the pressure drops instantly to the inlet pressure
This can be shown on a pressure-volume diagram (PV diagram) below.



The indicator (PV)diagram shown above represents the ideal case in which the air in the cylinder is considered a perfect gas and all processes are reversible.

In this ideal case the temperature is constant at T_1 for the induction stroke 4 to 1.
Line 123 represents the compression and delivery stroke.
It is assumed that the compression from 1 to 2 is reversible polytropic according to the law $pV^n = C$.

The delivery takes place along the line 2 to 3, again the temperature remaining constant at the value of T_2 which in turn depends upon the law of compression.

The net work done in the cycle is given by the net area of the *p*- *V* diagram

Indicated work done on the gas per cycle = area 12341

Area 12341= area12YZ + area 23XY - area 14XZ

Area 12YZ is the area under the poly-tropic compression curve $= \dfrac{p_1V_1 - p_2V_2}{n-1}$

Area 23XY is the flow work required to deliver the air $= p_2(V_3 - V_2)$

Area 14XZ is the work done by the air during the induction stroke $= p_1(V_1 - V_4)$

We can add these equations to obtain the net work and use pV = mRT to give

$$\text{Work input per cycle} = \frac{n}{n-1}(P_2V_2 - p_1V_1) = \frac{n}{n-1}mR(T_2 - T_1)$$

Where "m" is the mass induced and delivered per cycle.
If the mass flow rate is used then indicated power will be given

If we use the relationship $T_2 = T_1\left(\dfrac{p_2}{p_1}\right)^{\frac{n-1}{n}}$ the equation above can be further modified

$$\text{Indicated power} = \frac{n}{n-1}p_1\dot{V}_n\left\{\left(\frac{p_2}{p_1}\right)^{\frac{n-1}{n}} - 1\right\} \quad \text{Where } \dot{\mathrm{V}}_{\mathrm{n}} = \text{induced volume per unit time}$$

$$\text{Indicated power} = \frac{n}{n-1}\dot{m}RT_1\left\{\left(\frac{p_2}{p_1}\right)^{\frac{n-1}{n}} - 1\right\} \quad \text{Where } \dot{\mathrm{m}} = \text{induced mass per unit time}$$

This is the work required to compress the air it does not take into account any losses in the drive train.
The following relationships take account of this, whatever power you end up using is dictated by the information given in the problem you are solving.

$$\text{Shaft power} = \text{indicated power} + \text{friction power}$$

$$\text{Mechanical efficiency} = \frac{\text{indicated power}}{\text{Shaft power}}$$

$$\text{Input power} = \frac{\text{shaft power}}{\text{efficiency of motor drive}}$$

Example

A reciprocating air compressor without clearance runs at 5 rev/sec. It takes in 90 m^3/h of air at 1 bar and delivers it at 10 bar.
The index of compression is 1.3.
Calculate

The work transfer per cycle.

In this case we need the volume flow rate par cycle to use in the equation

$$\text{Work transfer} = \frac{n}{n-1} p_1 V_{cyl} \left\{ \left(\frac{p_2}{p_1} \right)^{\frac{n-1}{n}} - 1 \right\}$$

$$\text{Where } \mathrm{V_{cyl}} = \text{induced volume per cycle}$$

$$\mathrm{V_{cyl}} = \frac{\text{volume flow}}{\text{Cycles/unit time}}$$

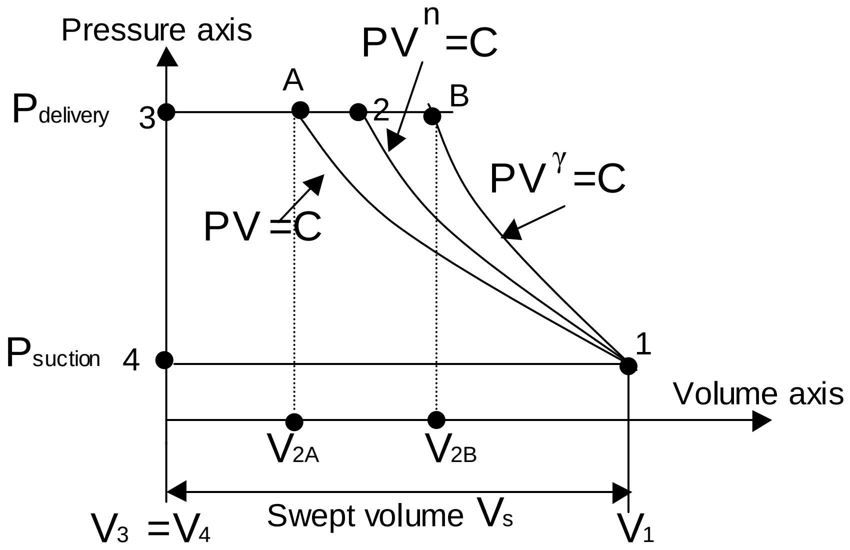
$$\mathrm{V_{cyl}} = \frac{90}{3600 \times 5} = 0.005 \frac{m^3}{cycle}$$

$$\text{Work transfer} = \frac{1.3}{1.3-1} \times 1 \times 10^5 \times 0.005 \left\{ \left(\frac{10}{1} \right)^{\frac{1..3-1}{1.3}} - 1 \right\}$$

$$\text{Work transfer} = 1.519 kJ / cycle$$

The condition for minimum work

The work done on the gas is given by the area of the indicator diagram will be a minimum when the area of the diagram is a minimum.
The height of the diagram is fixed by the required pressure ratio.
The length of the diagram is fixed by the swept volume.
The only process which can influence the area of the diagram is the compression curve, which is determined by the index of compression.



This diagram shows the limits of the possible values of n which also shows that V_2 can vary from a maximum of V_{2B} to a minimum of V_{2A}.

Line 1 to A is Isothermal pV = constant
Line 1 to 2 is Polytropic pV^n = constant
Line 1 to B is Isentropic pV^γ = constant

All processes are reversible.

The indicated work done when the gas is compressed isothermally is given by the area 1A341.
Using the same process as before we find that the net work is that due to the isothermal work.

$$\text{Isothermal work per cycle} = p_1 V_1 \ln\left(\frac{p_2}{p_1}\right)$$

$$\text{Isothermal power} = p_1 \dot{V}_a \ln\left(\frac{p_2}{p_1}\right) \quad \text{where } \dot{V}_a \text{ is the induced volume per unit time}$$

$$\text{Isothermal power} = \dot{m} R T \ln\left(\frac{p_2}{p_1}\right) \quad \text{where } \dot{m} \text{ is the induced mass per unit time}$$

For a reciprocating compressor a comparison between the actual work and the minimum or isothermal work is made by means of an isothermal efficiency.
Thus the higher the isothermal efficiency the closer the actual work has approached the ideal minimum.

$$\text{Isothermal efficiency} = \frac{\text{Isothermal work}}{\text{actual work}} = \frac{\text{Isothermal power}}{\text{actual power}}$$

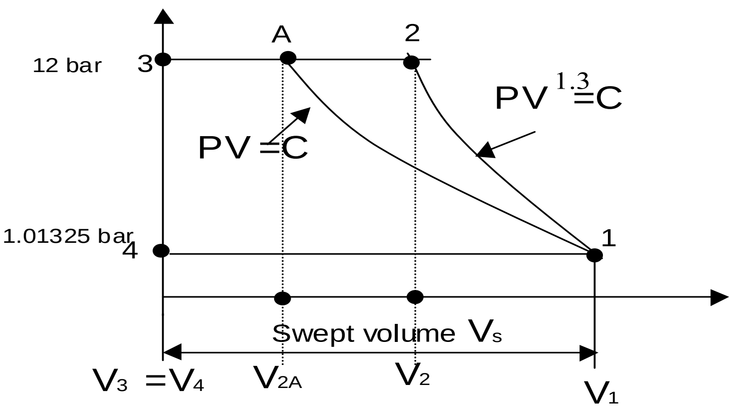
Isothermal compression is the most desirable compression process and gives the minimum work to be done on the gas.
The least desirable form of compression in reciprocating compressors is that given by the isentropic process.
The actual form of compression will usually be between these two limits with the value of n usually between 1.2 and 1.3.

This means that in an actual compressor the gas temperature must be kept as close as possible to its initial value.
The main method used for cooling the air is by surrounding the cylinder with a water jacket and designing for the best ratio of surface area to volume of the cylinder.

Example compressor without clearance

A single cylinder single stage air compressor takes in 1kg of air at NTP and compresses it to 12 bar according to the law $pV^{1.3}$ = constant.
For air R = 287 J/kgK and $\gamma = 1.4$
Calculate

a) The actual work transfer
b) The isothermal efficiency



This diagram shows the two conditions, The actual compression 1 to 2 and that for the isothermal compression 1 to A.
We will use the basic equation for net work and also determine the work from first principles to prove that they are the same.

Using the equation developed previously

$$\text{Indicated work transfer} = \frac{n}{n-1}\dot{m}RT_1\left\{\left(\frac{p_2}{p_1}\right)^{\frac{n-1}{n}} - 1\right\}$$

$$\text{Indicated work transfer} = \frac{1.3}{1.3-1}1\times 287\times 288\left\{\left(\frac{12}{1.01325}\right)^{\frac{1.3-1}{1.3}} - 1\right\}$$

$$\text{Indicated work transfer} = 275.43kJ$$

This is a very straight forward procedure, however calculating this from first principles allows us to appreciate what is happening in the cycle.
To calculate this from first principles we will need to obtain the volumes at each point for the 1 kg of air being delivered and then calculate the net area of the diagram. Remember the mass flow is constant, there is no clearance and all volumes are measured from the pressure axis.

From $pV = mRT$

$$V_1 = \frac{mRT_1}{P} \qquad V_1 = \frac{1\times 287\times 288}{1.01325\times 10^5} = 0.81575m^3$$

$$V_2 = V_1\left(\frac{p_2}{p_1}\right)^{\frac{1}{n}} \qquad V_2 = 0.81575\left(\frac{1.01325}{12}\right)^{\frac{1}{1.3}} = 0.1218m^3$$

$$\text{For no clearance } V_3 = V_4 = 0$$

$$\text{Polytropic compression} = \frac{p_1V_1 - p_2V_2}{n-1}$$

$$= \frac{(1.01325\times 10^5\times 0.81575 - 12\times 10^5\times 0.1218)}{1.3-1}$$

$$\text{Polytropic compression work transfer} = -211.68kJ$$

$$\text{Delivery flow work} = p_2(V_3 - V_2) = 12\times 10^5(0 - 0.1218) = -146.16kJ$$

$$\text{Induction flow work} = p_1(V_1 - V_4) = 1.01325\times 10^5(0.81575 - 0) = 82.655kJ$$

$$\text{Net Work} = (-211.68) + (-146.16) + 82.655 = -275.185kJ$$

This is very close to the original value obtained using the single equation which was expected.

We now need to determine the isothermal work.
This is obtained by calculating the net area of the pv diagram in the same way as that above.

Net area = isothermal work + delivery flow work + induction flow work

$$\text{Isothermal work} = p_1 V_1 \ln\left(\frac{p_1}{p_2}\right) = 1.01325 \times 10^5 \times 0.81575 \ln\left(\frac{1.01325}{12}\right)$$

$$\text{Isothermal work} = -204.3kJ$$

The induction flow work is the same as the first part = + 82.655 kJ

The delivery flow work requires us to calculate a new volume due to isothermal compression.

$$p_2 V_2 = p_1 V_1$$

$$V_2 = V_1\left(\frac{p_1}{p_2}\right)$$

$$V_{2isothermal} = 0.81575\left(\frac{1.01325}{12}\right) = 0.0688m^3$$

$$\text{Delivery flow work} = p_2\left(V_3 - V_2\right) = 12 \times 10^5\left(0 - 06888\right) = -82.655kJ$$

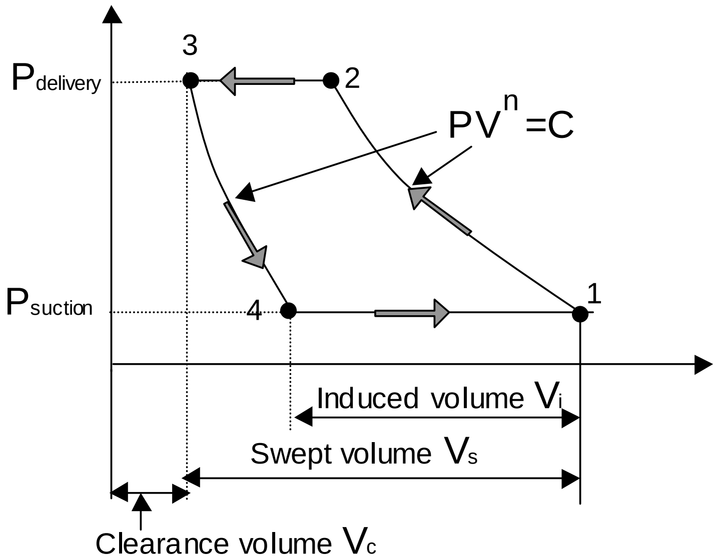
This cancels out the induction work hence the net work is only that due to the isothermal compression, which shows that for this compressor without clearance the minimum work is that due only to the isothermal process.

The isothermal efficiency is obtained from the equation given above

$$\text{Isothermal efficiency} = \frac{\text{Isothermal work}}{\text{actual work}} = \frac{204.3}{275.4} = 74\%$$

Reciprocating compressors including clearance

Clearance is necessary in a compressor to give mechanical freedom to working parts and allow the necessary space for valve operations.
The diagram shows an ideal indicator diagram with the clearance included.
The clearance volume can range from about 2% to around 16% of the swept volume .



In this case when the piston is at top dead centre the clearance volume V_c is full of gas at pressure P_2 and temperature T_2.
As the piston moves towards bottom dead centre on the induction stroke the air in the clearance volume expands until the suction pressure is reached, only at this point can air enter the cylinder.
The effect of this clearance is to reduce the amount of air induced from the swept volume to $V_1 - V_4$

The net work is obtained by calculating the area under the diagram in the same way as before, the only difference being the extra work produced by the air expanding from the delivery pressure down to the suction pressure.

The masses of gas at the four principal points on the diagram above are $m_1 = m_2$ and $m_3 = m_4$
The mass delivered is given by $(m_2 - m_3)$.
This is equal to mass induced, given by $(m_1 - m_4)$.

The indicated work done is given by the net area of the *p-V* diagram which is obtained in the same way as for the compressor without clearance.
In this case with clearance, the work of the gas expanding from 3 to 4 cancels out the work in compressing this portion of the gas providing that the expansion index is the same as the compression index.
Hence the work done per unit mass of air delivered is unaffected by the size of the clearance volume.

The indicated work is calculated from the same equation as previously obtained only in this case the induced mass or volume is used, again if volume and mass flow rates are used, power rather than work per cycle is used.

$$\text{Indicated work} = \frac{n}{n-1} Rm(T_2 - T_1)$$

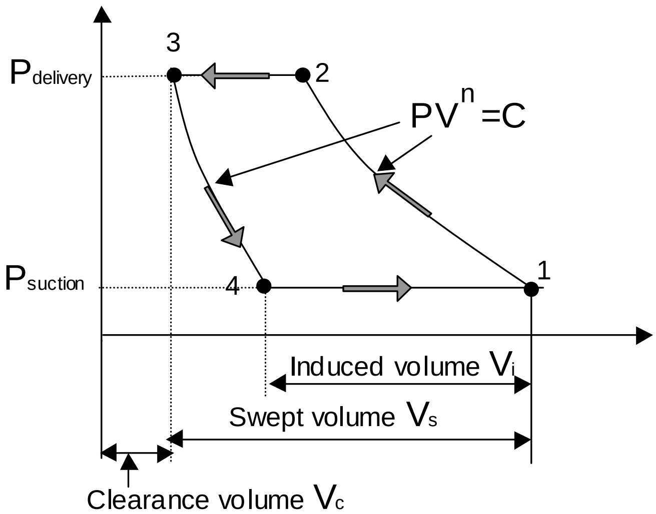
$$\text{Work transfer} = \frac{n}{n-1} p_1 V_{induced} \left\{ \left(\frac{p_2}{p_1} \right)^{\frac{n-1}{n}} - 1 \right\}$$

$$\text{Work transfer} = \frac{n}{n-1} m_{induced} RT_1 \left\{ \left(\frac{p_2}{p_1} \right)^{\frac{n-1}{n}} - 1 \right\}$$

Example compressor with clearance

A single cylinder single stage air compressor takes in 1kg of air at NTP and compresses it to 12 bar.
Compression and expansion processes take place to the law $pV^{1.3}$ = constant.
The clearance volume is 1% of the swept volume.
For air R = 287 J/kgK and γ = 1.4
Calculate

a) The actual work transfer
b) The heat transfer during compression



For 1 kg of air delivered at NTP $\qquad pV = mRT$

$$V_{induced} = \frac{1 \times 287 \times 288}{1.01325 \times 10^5} = 0.81575 m^3$$

$$\text{Work transfer} = \frac{n}{n-1} p_1 V_{induced} \left\{ \left(\frac{p_2}{p_1} \right)^{\frac{n-1}{n}} - 1 \right\}$$

$$\text{Work transfer} = \frac{1.3}{1.3-1} 1.01325 \times 10^5 \times 0.81575 \left\{ \left(\frac{12}{1.01325} \right)^{\frac{1.3-1}{1.3}} - 1 \right\}$$

$$\text{Work transfer} = 275.429 kJ$$

Note that equation gives a positive value even though the work is being done on the air that is a work input to the system which by convention would be negative.

To calculate this from first principles requires the volumes to be calculated, this is done by obtaining the induced volume in terms of the swept volume and solving for the swept volume.

The induced volume is $V_1 - V_4 = 0.8157\ m^3$.

$V_1 = V_{swept} + V_{clearance}$, $\quad V_1 = V_s + 0.01V_s$,

$$V_1 = 1.01V_s$$
$$V_3 = 0.01V_s$$

$$V_4 = 0.01V_s \left(\frac{p_3}{p_4} \right)^{\frac{1}{n}}$$

$$V_4 = 0.01V_s \left(\frac{1.2}{1.01325} \right)^{\frac{1}{1.3}} = 0.0669 V_s m^3$$

$$V_{induced} = V_1 - V_4 = 1.01V_s - 0.0669V_s$$

$$0.81575 = 0.943V_s$$

$$V_s = 0.865 m^3$$

The volumes are therefore

$$V_1 = 0.8736 m^3 \quad V_2 = 0.1305 m^3 \quad V_3 = 0.00865 m^3 \quad V_s = 0.0579 m^3$$

We can now use these volumes to determine the area under the curve.

$$\text{Compression Work} = \frac{p_1 v_1 - p_2 v_2}{\text{n - 1}} + p_2(v_3 - v_2)$$

$$= \frac{(1.01325 \times 10^5 \times 0.87366) - (12 \times 10^5 \times 0.1305)}{1.3 - 1} + 12 \times 10^5 (0.00856 - 0.1305)$$

$$= (-226.881) + (-146.205)$$

$$= -373.089 kJ$$

$$\text{Expansion Work} = \frac{p_3 v_3 - p_4 v_4}{\text{n - 1}} + p_1(v_1 - v_4)$$

$$= \frac{(12 \times 10^5 \times 0.00865) - (1.01324 \times 10^5 \times 0.0579)}{1.3 - 1} + 1.0132510^5 (0.81575)$$

$$= 15.044 + 87.646$$

$$= 97.699 kJ$$

$$\text{Net Work} = (-373.089) + 97.699 = -275.39 kJ$$

This is the same result as before only the negative sign is given indicating that the work has been done on the gas.

The value obtained is the actual work transfer for the cycle, it takes account of the fact that the compression work carried out on the gas in the clearance volume is recovered during its expansion process.

The question also asks for the heat transfer during the compression process, to do this we can apply the non flow energy equation to this process

The question is based on 1 kg of air delivered this means that the actual amount of air being compressed is slightly greater.
Thus using V_1, the mass at point 1 is given by

$$pV = mRT$$

$$m = \frac{1.01325 \times 10^5 \times 0.8736}{287 \times 288} = 1.071 kg$$

The increase in temperature from 1 to 2 is given by

$$T_2 = T_1\left(\frac{P_2}{P_1}\right)^{\frac{n-1}{n}} = 288\left(\frac{12}{1.01325}\right)^{\frac{1.3-1}{1.3}} = 509.465K$$

If we apply the non flow energy equation to the compression process

$$Q = W + \Delta U$$

The internal energy is given by $\Delta U = mc_v(T_2 - T_1)$

$$\Delta U = 1.071 \times 717.5(509.465 - 288)$$

$$\Delta U = 170.181kJ$$

The polytropic work has been previously calculated using the volumes and was found to be -226.81 kJ

The internal energy has been increased by 170.181 kJ

$$Q = -226.81 + 170.181$$

$$Q = -56.628kJ$$

Which means that 56.628 kJ has been rejected to the surroundings during the polytropic compression process.
Note that it is important to obtain the correct value for the compression work and also apply the correct notation for the direction of energy transfers.

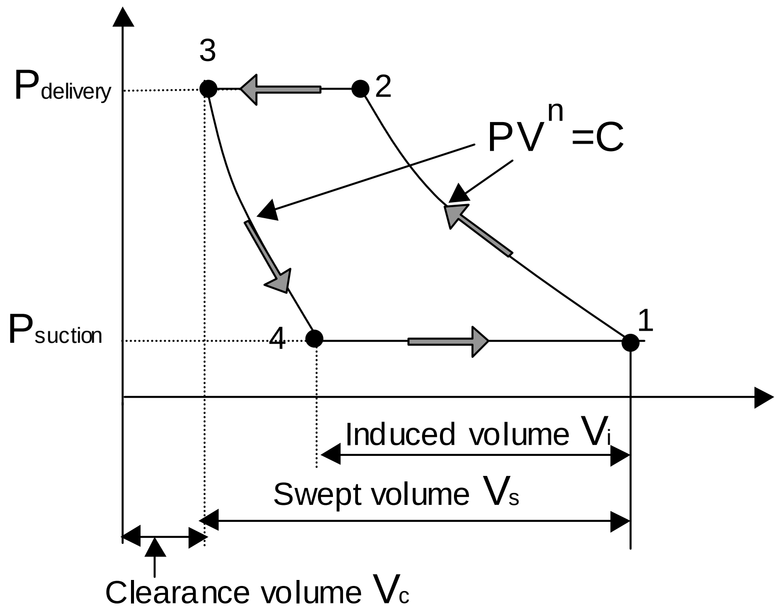
Volumetric efficiency, η_v

One of the effects of clearance is to reduce the induced volume to a value less than that of the swept volume.

This means that for a required induction the cylinder size must be increased over that calculated on the assumption of zero clearance.

The volume of air dealt with per unit time by an air compressor is quoted as the free air delivery (FAD), and is the delivered volume flow rate, measured at a reference pressure and temperature.

The mass delivered however will be the same as the mass induced regardless of pressure and temperature.



The volumetric efficiency is defined as follows:

$$\eta_v = \frac{\text{mass delivered}}{\text{mass in swept volume}}$$

$$pV = mRT \quad \text{and m} = \frac{\text{pV}}{\text{RT}}$$

Therefore when the pressure and temperatures are the same the volumetric efficiency becomes

$$\eta_v = \frac{\text{Volume induced}}{\text{swept volume}} = \frac{V_1 - V_4}{V_{swept}}$$

We can rearrange this expression in terms of the clearance volume and pressure ratio in the following manner.

Volume induced $\quad V_1 - V_4 = V_s + V_c - V_4$

but $\quad \dfrac{V_4}{V_3} = \left(\dfrac{p_2}{p_1}\right)^{\frac{1}{n}}$

Therefore the volume induced becomes

$$V_1 - V_4 = V_s + V_c - V_c\left(\frac{p_2}{p_1}\right)^{\frac{1}{n}}$$

If substitute this expression in $\quad \eta_v = \dfrac{V_1 - V_4}{V_s}$

We have an equation only in terms of the compressor swept volume , clearance volume, and pressure ratio.

$$\eta_v = 1 - \frac{V_c}{V_s}\left\{\left(\frac{p_2}{p_1}\right)^{\frac{1}{n}} - 1\right\}$$

This theoretical formula for volumetric efficiency shows that the Volumetric Efficiency decreases:

(1) As the clearance increases,
(2) As the compression ratio increases
(3) As n decreases.

Cylinder Clearance and Volumetric Efficiency

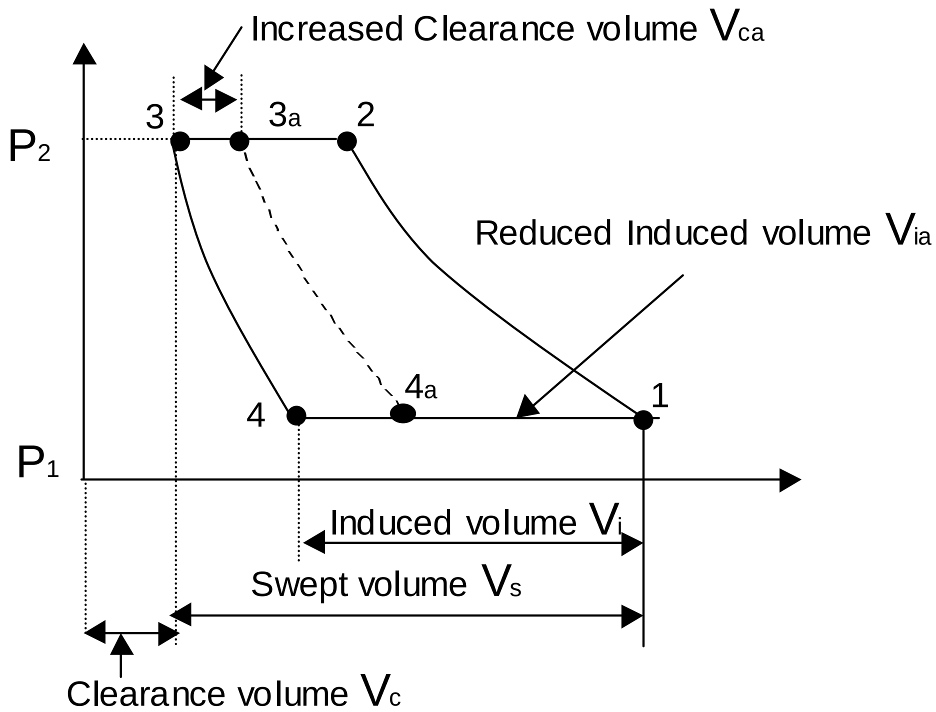
Cylinder clearance cannot be completely eliminated.
Normal clearance does not include clearance volume that may have been added for other purposes, such as capacity control.
Normal clearance variations have no effect on power requirements, however its effect on capacity should be understood because of the wide application of a variation in clearance volume for capacity control and other purposes.
Although clearance is always maintained at the lowest value consistent with adequate valving and running clearance it is more of a concern at higher compression ratios and when handling gases with low specific heat ratios,

AS THE CLEARANCE INCREASES,
The equation pV= mRT can be rearranged so that volume is proportional to mass providing that the pressure and temperatures are constant.

$$V = m\left(\frac{RT}{P}\right)$$

So for any condition of pressure and temperature as the mass in the clearance volume increases so the volume at the end of the expansion to inlet conditions increases and so the induced volume is reduced, the new cycle is shown as 1, 2, 3a, 4a, on the diagram below.
The swept volume is constant so as the induced volume is reduced the volumetric efficiency must fall.

$$\eta_v = \frac{\text{Volume induced}}{\text{swept volume}} = \frac{V_1 - V_4}{V_{swept}}$$

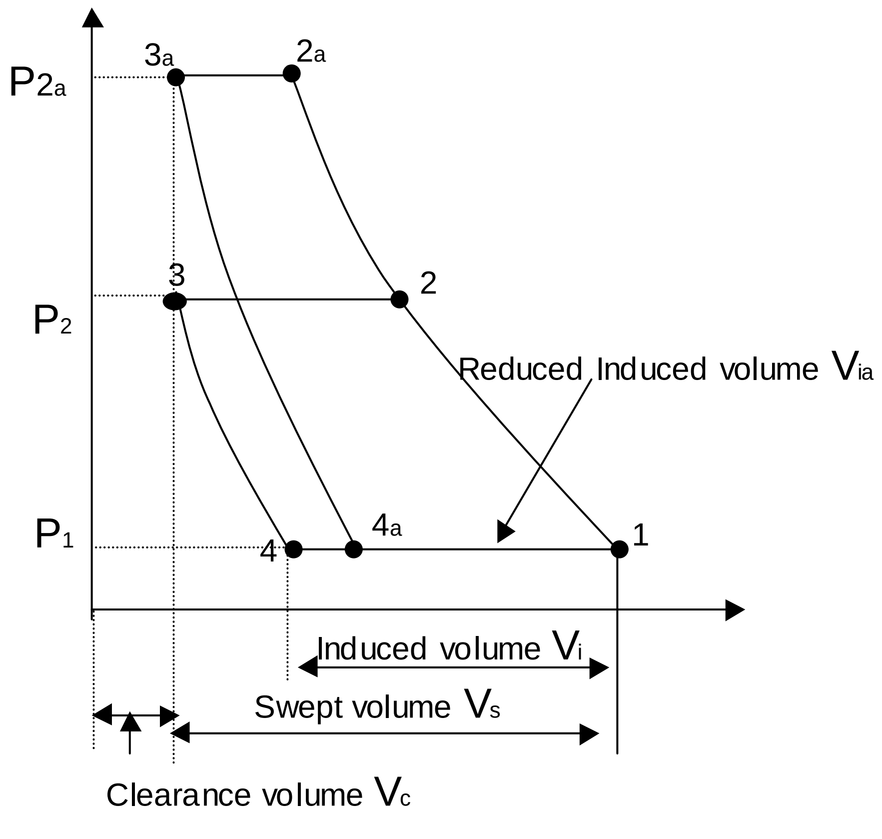


AS THE COMPRESSION RATIO INCREASES

In this particular case we can look at $p_3V_3^n = p_4V_4^n$

Rearranging this to make volume at point 4 the subject gives $V_4 = V_3\left(\frac{p_4}{p_3}\right)^{\frac{1}{n}}$

we can see that for a fixed clearance volume V_3 and index of expansion n, V_4 depends upon the pressure ratio and as this increases so the volume at 4 increases to 4a and the induced volume is reduced producing a fall in the volumetric efficiency.



AS "n" DECREASES

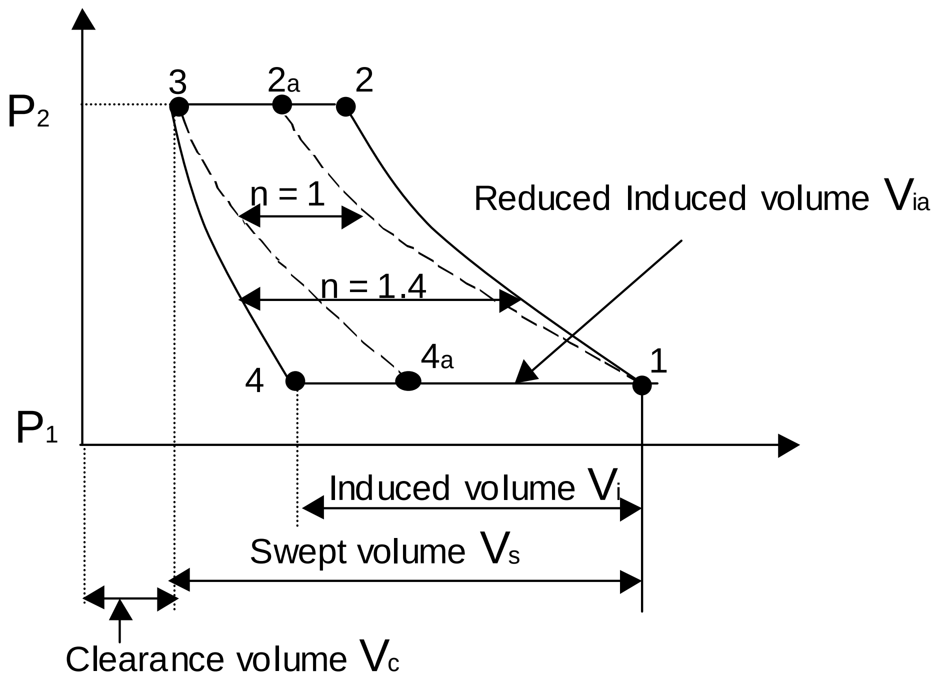
Again we can look at $p_3V_3^n = p_4V_4^n$

Rearranging this to make volume at point 4 the subject gives $V_4 = V_3\left(\frac{p_4}{p_3}\right)^{\frac{1}{n}}$

If the index n falls the value of 1/n is increased which increases the power of the compression ratio.

This is best seen by example
If n = 1 and the pressure ratio is 4 then 1/1=1 therefore V_3 is multiplied by 4 to the power 1 which is 4
If n = 1.4 and the pressure ratio is 4 then 1/1.4 = 0.714 therefore V_3 is multiplied by 4 to the power 0.714 which is 2.69.
Therefore when n = 1 $V_4 = 4\ V_3$ and when n = 1.4 $V_4 = 2.69V_3$

This is shown on the diagram below at points 4a and 4 respectively.
Thus the larger the value of V_4 the lower the induced volume and with it the volumetric efficiency



Multistage compression

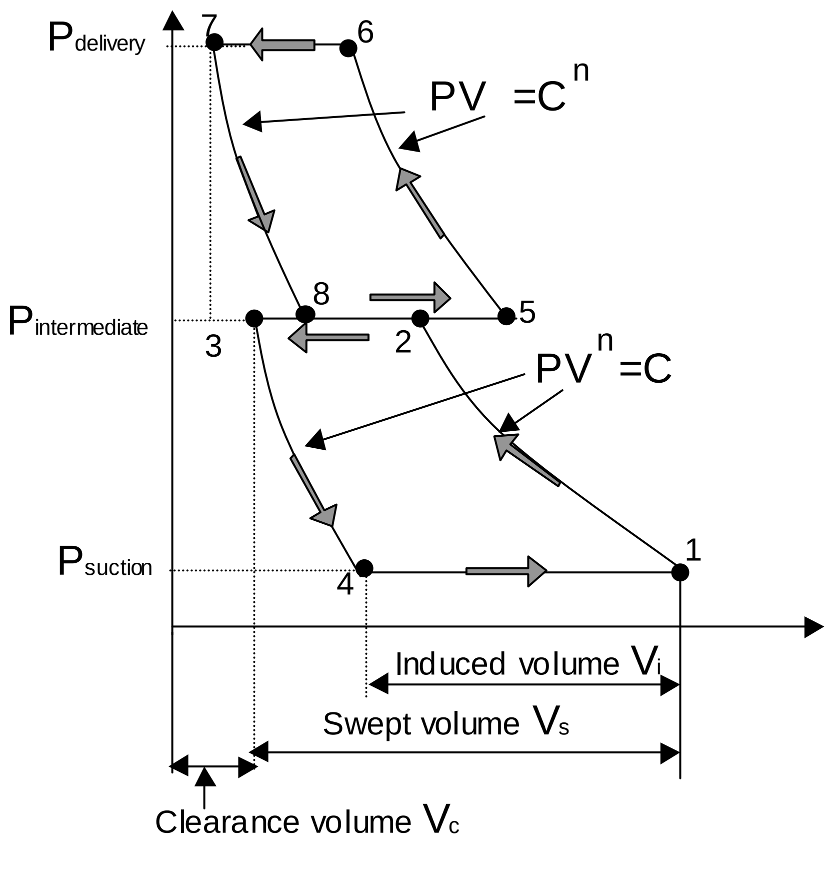
As we have just seen from the equation $\eta_v = 1 - \frac{V_c}{V_s}\left\{\left(\frac{p_2}{p_1}\right)^{\frac{1}{n}} - 1\right\}$

as the pressure ratio increases the volumetric efficiency decreases.

The temperature after compression is given by equation $T_2 = T_1\left(\frac{p_2}{p_1}\right)^{\frac{n-1}{n}}$

Therefore delivery temperature also increases with the pressure ratio.
Thus for a required Free Air Delivery the cylinder size would have to increase as the pressure ratio increases.

The volumetric efficiency can be improved by carrying out the compression in two or more stages.
After the first stage of compression the gas is passed into smaller cylinder in which the gas is compressed to the required final pressure if the machine has two stages, but it could be delivered to a third cylinder if higher pressures were required.
The cylinders of the successive stages are proportioned to take the volume of gas delivered from the previous stage.
The indicator diagram for a two-stage machine is shown below.



In this diagram it is assumed that the delivery process from the first or LP stage and the induction process of the second or HP stage are at the same pressure.
The low pressure cycle is defined as 1, 2, 3, 4.
The high pressure cycle is defined as 5, 6, 7, 8.
The index of compression and expansion is the same for both stages.

We can use all the previous equations we have developed and aply them to each stage.
So for a two stage machine using the above diagram.

Total work = LP stage work + HP stage work.

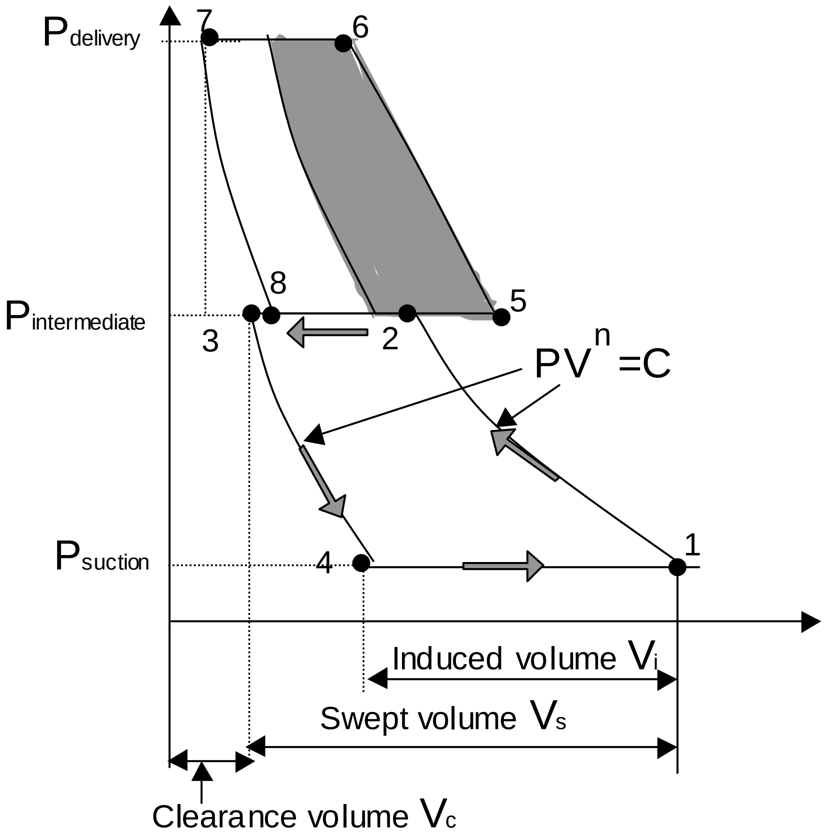
$$\text{Total Work} = \frac{n}{n-1} mRT_1 \left\{ \left(\frac{p_2}{p_1} \right)^{\frac{n-1}{n}} - 1 \right\} + \frac{n}{n-1} mRT_5 \left\{ \left(\frac{p_6}{p_5} \right)^{\frac{n-1}{n}} - 1 \right\}$$

As before if we use mass per cycle we have the work transfer per cycle or if we use a mass flow rate we have the power required to compress the air.
We could also use the induced volume and pressure for each stage but care is needed to make sure the correct values are used.
We can then use efficiencies if given in the question to determine the input power.
The ideal isothermal compression can only be obtained if perfect cooling is continuous however this is difficult to obtain during normal compression.

With multistage compression there is the possibility of the gas being cooled as it is transferred from one stage to the next, by passing it through an inter-cooler.
If inter-cooling is complete, the gas will enter the second stage at the same temperature at which it entered the first stage.
The relationship below shows that if the temperature of the air is reduced then so is the volume since the mass and pressure are constant.

$$V = T\left(\frac{mR}{P} \right)$$

If the volume is reduced then so is the area of the PV diagram hence there will be a reduction in work, this saving is shown by the shaded area on the diagram below.



The two indicator diagrams 1234 and 5678 are shown with a common pressure, $p_{intermediate}$. This does not occur in a real machine as there is a small pressure drop between the cylinders however this can be ignored in test examples.
An after-cooler can also be fitted after the delivery process to cool the gas.

The delivery temperatures from the two stages are given by

$$T_2 = T_1\left(\frac{p_2}{p_1}\right)^{\frac{n-1}{n}} \qquad \text{and} \qquad T_6 = T_5\left(\frac{p_6}{p_5}\right)^{\frac{n-1}{n}}$$

respectively.

If the gas is cooled in the inter-cooler back to inlet temperature, T_5 would equal T_1 this situation is called complete inter-cooling.
To calculate the indicated power the previous equations can be applied to each stage separately and the results added together.

The ideal intermediate pressure

The intermediate pressure p_2 which is the same as p_5 and intestage temperature T_5, have a direct effect on the work to be done on the gas in each stage.
If we consider a two stage machine

Total work = LP stage work + HP stage work.

$$\text{Total work} = \frac{n}{n-1} mRT_1 \left\{ \left(\frac{p_2}{p_1} \right)^{\frac{n-1}{n}} - 1 \right\} + \frac{n}{n-1} mRT_5 \left\{ \left(\frac{p_6}{p_5} \right)^{\frac{n-1}{n}} - 1 \right\}$$

If inter-cooling is complete the temperature at the start of each stage is T_1.

$$\text{Total work} = \frac{n}{n-1} mRT_1 \left\{ \left(\frac{p_2}{p_1} \right)^{\frac{n-1}{n}} - 1 + \left(\frac{p_6}{p_2} \right)^{\frac{n-1}{n}} - 1 \right\}$$

If p_1, T_1, and p_6 are fixed, then the optimum value of p_2 which makes the work a minimum can be obtained by equating d(work)/(dp_2) to zero, i.e. optimum value of p_2 when

$$\frac{d}{dp_2} \left\{ \left(\frac{p_2}{p_1} \right)^{\frac{n-1}{n}} + \left(\frac{p_6}{p_2} \right)^{\frac{n-1}{n}} - 2 \right\} = 0 \qquad \frac{n}{n-1} mRT_1 \text{ are constant}$$

This gives $P_2^2 = P_1 P_6$ or $\frac{p_2}{p_1} = \frac{p_6}{p_2}$ the pressure ratio is the same for each stage.

Let the pressure ratio per stage equal r_p

Then $\frac{p_2}{p_1} = r_p \qquad p_2 = r_p p_1$ and $\frac{p_6}{p_2} = r_p \qquad p_6 = r_p p_2$

Sub for p_2 in the p_6 equation $\quad p_6 = r_p (r_p p_1) = r_p^2 p_1 \quad$ Therefore $\quad r_p = \left(\frac{p_6}{p_1} \right)^{\frac{1}{2}}$

This gives the pressure ratio per stage as the square root of the overall pressure ratio of the two stage machine.
This can be extended for any number of stages say to give

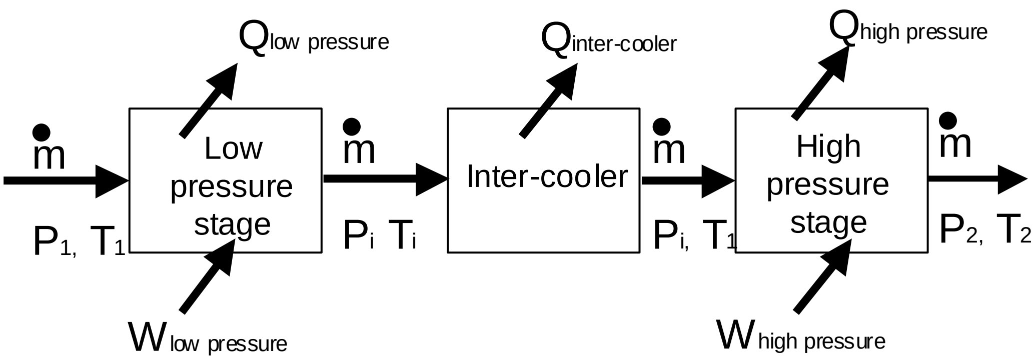
$$\text{Pressure ratio per stage } r_p = \left(\frac{p_{delivery}}{p_{induction}} \right)^{\frac{1}{\text{number of stages}}}$$

$$\text{Total minimum work} = \text{number of stages} \times \frac{n}{n-1} mRT_1 \left\{ \left(\frac{p_2}{p_1} \right)^{\frac{n-1}{n}} - 1 \right\}$$

Hence for a multi-stage machine the condition for minimum work is that the pressure ratio in each stage is the same and that inter-cooling is complete.
Remember as before $PV_{induced}$ may be used instead of mRT_1 and if mass or volume flow rates are used instead of per cycle then power is obtained.

If in doubt allways put in the units of each term.

Energy balance for a two-stage machine with inter-cooler



The above diagram shows steady flow through a two stage-reciprocating compressor with inter-cooling.
The steady-flow energy equation can be applied to the low pressure stage, the inter-cooler, and the high pressure stage.
The mass flow rate is constant through the system.
Changes in kinetic energy and height can be neglected,

Steady flow energy equation becomes $$h_1 + Q = W + h_2$$

And for a perfect gas $$h = c_p T \quad \text{and } \mathrm{H} = mc_p T$$

For the low pressure stage stage, $$mc_p T_1 + Q_l = W_l + mc_p T_i$$

Therefore Heat rejected in low pressure stage $$Q_l = -W_l + mc_p (T_i - T_1)$$

This must not be confused with the heat rejected due to the polytropic compression of the gas which is a non flow process and was dealt with earlier.
This is the heat rejected for the entire stage due to the flow process.

For the inter-cooler, $$mc_p T_i + Q_l = mc_p T_1$$

Therefore Heat rejected in inter-cooler $$Q_I = mc_p (T_1 - T_i)$$

For the high pressure stage, $$mc_p T_1 + Q_H = W_H + mc_p T_2$$

Therefore Heat rejected in high pressure stage $$Q_H = -W_H + mc_p (T_2 - T_1)$$

Again,this must not be confused with the heat rejected due to the polytropic compression of the gas which is a non flow process and was dealt with earlier.
This is the heat rejected for the entire stage due to the flow process.

The question below is the previous example of a compressor with clearance.
We have already calculated the net work transfer and the heat rejected during the compression process.
We can apply a steady flow analysis to determine the heat lost in the stage.

A single cylinder single stage air compressor takes in 1kg of air at NTP and compresses it to 12 bar.
Compression and expansion processes take place to the law $pV^{1.3}$ = constant.
The clearance volume is 1% of the swept volume.
For air R = 287 J/kgK and $\gamma = 1.4$

The net work was previously found to be -275.389 kJ
From the steady flow energy equation

$$H_1 + Q = H_2 + W$$

$$\Delta H = mc_p(T_2 - T_1) = 1 \times 1.0045(509.465 - 288)$$

$$\Delta H = 222.46kJ$$

$$Q = \Delta H + W$$

$$Q = 222.47 + (-275.389)$$

$$Q = 52.919kJ$$

We can note then that the net heat transfer is different in both cases and we must be clear about what any question is asking us to find either heat rejection during compression which is clearly a non flow polytropic process, or heat rejected during the stage which would take account of the fact that overall the delivery is a steady flow process.

Worked examples one

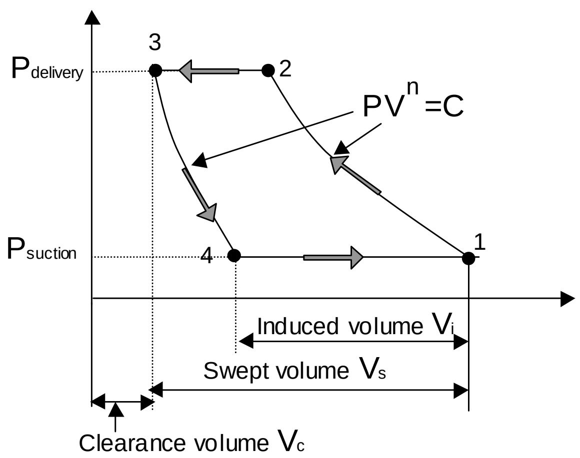
A single stage reciprocating air compressor has a free air delivery of 3 m^3/min at the inlet conditions of 1.01 bar 15°C.

The pressure ratio is 10 and the clearance is 6 % of the swept volume which is 14.2 litres.

Calculate

a) The compressor speed.

b) The power absorbed in compressing the air.



To obtain the compressor speed we need to calculate the amount of air delivered per cycle which is the induced volume V_1- V_4.

$V_1 = V_{swept} + V_{clearance}$, $V_1 = V_s + 0.06V_s$, $V_1 = 1.06V_s$

$$V_1 = 1.06 \text{ x } 14.2 = 15.052 \text{ litres}$$

$$V_1 = 0.015052 \text{ m}^3\text{/ cylinder}$$

$V_3 = V_c = 0.06V_s = 0.06 \text{ x } 14.2 = 0.852 \text{ litres} = 0.00852 \text{ m}^3$

$$p_4V_4 = p_3V_3 \qquad V_4 = V_3\left(\frac{p_3}{p_4}\right)^{\frac{1}{n}} \qquad V_4 = 0.00852(10)^{\frac{1}{1..3}} \qquad V_4 = 5x10^{-3}m^3$$

$$V_{induced} = V_1 - V_4 = 0.01502 - 0.005$$

$$V_{induced} = V_1 - V_4 = 0.01\frac{m^3}{cycle}$$

$$\text{speed} = \frac{\text{free air delivered per minute}}{\text{air delivered per cycle}} = \frac{\text{cycles}}{\text{minute}}$$

$$\text{speed} = \frac{3}{0.01} = 300\frac{\text{revs}}{\text{minute}}$$

$$\text{Work transfer} = \frac{n}{n-1} p_1 V_{induced} \left\{ \left(\frac{p_2}{p_1} \right)^{\frac{n-1}{n}} - 1 \right\}$$

$$\text{Work transfer} = \frac{1.3}{1.3-1} 1 \times 10^5 \times 0.01 \left\{ (10)^{\frac{1.3-1}{1.3}} - 1 \right\}$$

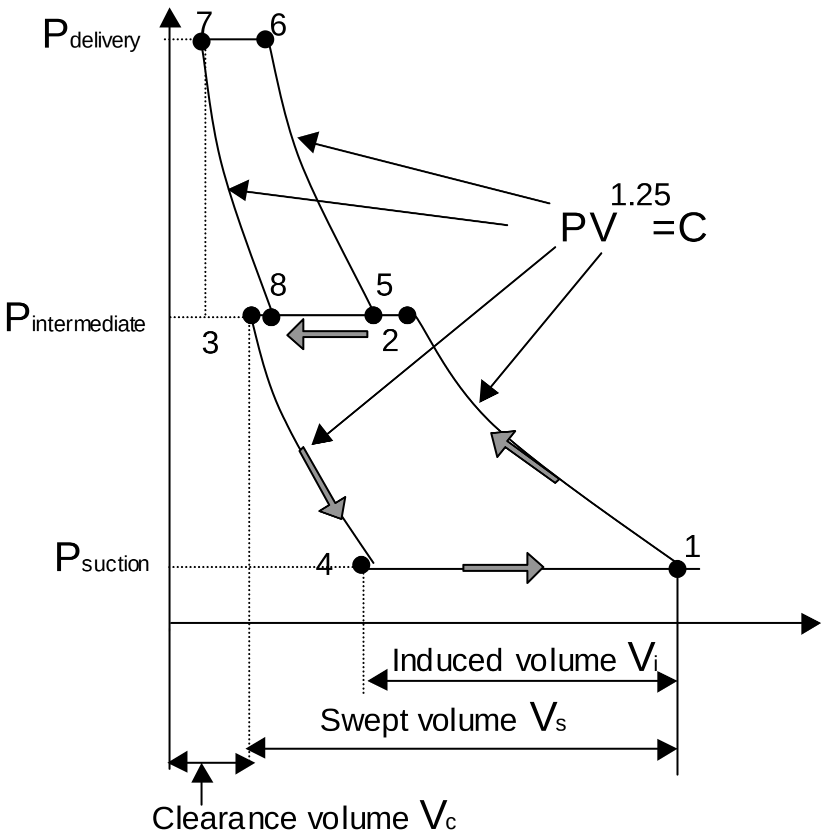
$$\text{Work transfer} = 3.038 kJ / cycle$$

$$\text{Power} = \text{Work per cycle} \times \text{cycles per unit time}$$

$$\text{Power} = 3.038 \times \left(\frac{300}{60} \right) = 15.19 \text{ kW}$$

Worked example two
A two stage inter-cooled single acting air compressor running at 300 rev/min delivers air at 9 bar with a mechanical efficiency of 80 %.
The low pressure bore is 250 mm with a stroke of 160 mm.
The clearance volume is 1 % of the swept volume.
First stage inlet conditions are 0.9 bar 20°C, while second stage inlet is at 3 bar 20°C.
The index of expansion and compression in both stages is 1.25.
For air r = 287 J/kgK
Calculate

a) The free air delivered at STP
b) The input power required to drive the compressor



The free air delivered is the volume of the mass induced calculated at standard temperatures and pressures.
To calculate the induced volume we can use the procedure shown in the previous examples

V_{swept} = cylinder area x stroke = $\pi/4D^2$ x stroke $=0.7854 \times 0.25^2 \times 0.16$

$$V_{swept} = 7.854\text{x}10^{-3}\ m^3$$

$V_1 = V_{swept} + V_{clearance}$, $V_1 = V_s + 0.01V_s$,

$$V_1 = 1.01V_s$$

$V_1 = 1.01 \times 7.854\text{x}10^{-3}\ m^3$ $V_1 = 7.92 \times 10^{-3}\ m^3$/ cycle

$V_3 = V_c = 0.01V_s$ $V_3 = 0.01 \times 7.854\text{x}10^{-3}$ $V_3 = 7.854\text{x}10^{-5} m^3$/cycle

$$p_4V_4^n = p_3V_3^n \qquad V_4 = V_3\left(\frac{p_3}{p_4}\right)^{\frac{1}{n}}$$

$$V_4 = 7.854\times10^{-5}\left(\frac{3}{0.9}\right)^{\frac{1}{1.25}} \qquad V_4 = 7.854\times10^{-5}(2.62)$$

$$V_4 = 2.057\times10^{-4}\ \text{m}^3 \text{ per cycle}$$

Volume induced = $V_1 - V_4$

Volume induced = $7.92 \times 10^{-3} - 2.057 \times 10^{-4} = 7.71 \times 10^{-3}$ m^3/cycle

To find the mass induced mass we can use the equation $pV = mRT$ where the volume is the induced volume.

Rearranging this equation gives $m = \dfrac{pV}{RT}$

$$m = \frac{0.9\times10^5\times7.71\times10^{-3}}{287\times293} = 8.25\times10^{-3}\ {}^{kg}\!/_{cycle}$$

mass flow rate is found by multiplying the mass per cycle by the number of cycles per unit time

$$\text{mass flow} = 8.25\times10^{-3}\times\frac{300}{60} = 0.0412\ {}^{\text{kg}}\!/_{\text{second}}$$

We can convert this mass flow to a volume flow at STP using $pV = mRT$ again

Therefore $V = \dfrac{mRT}{p}$ and $V = \dfrac{0.0412\times287\times273}{1.01325\times10^5} = 0.0319\ {}^{\text{m}^3}\!/_{\text{second}}$

Free air delivery is usually given in cubic meters per minute thus for this example the volume of air delivered at standard temperatures and pressure is

Free air delivery = 1.914 m^3/min

The power required to compress the air is given by multiplying the power to drive the compressor by the mechanical efficiency.

Power to compress the air is obtained by applying the equation for work to each stage. Note that although the inlet temperatures are the same for both stages the pressure ratios are not therefore the work per stage will be different.

For each stage $$Power = \frac{n}{n-1}\left(\dot{m}RT_{inlet}\right)\left[\left(\frac{p_{outlet}}{p_{inlet}}\right)^{\frac{n-1}{n}} - 1\right]$$

Total power is the first stage power plus the second stage power

$$\text{Total Power} = \frac{n}{n-1}\left(\dot{m}RT_1\right)\left[\left(\frac{p_2}{p_1}\right)^{\frac{n-1}{n}} - 1\right] + \frac{n}{n-1}\left(\dot{m}RT_5\right)\left[\left(\frac{p_6}{p_5}\right)^{\frac{n-1}{n}} - 1\right]$$

$$\text{Total Power} = \frac{1.25}{1.25-1}\left(0.0412 \times 287 \times 293\right)\left[\left[\left(\frac{3}{0.9}\right)^{\frac{1.25-1}{1.25}} - 1\right] + \left[\left(\frac{12}{3}\right)^{\frac{1.25-1}{1.25}} - 1\right]\right]$$

$$\text{Total Power} = 17322.7\left[\left[1.27 - 1\right] + \left[1.32 - 1\right]\right]$$

$$\text{Total Power} = 8.973 \ \text{kW}$$

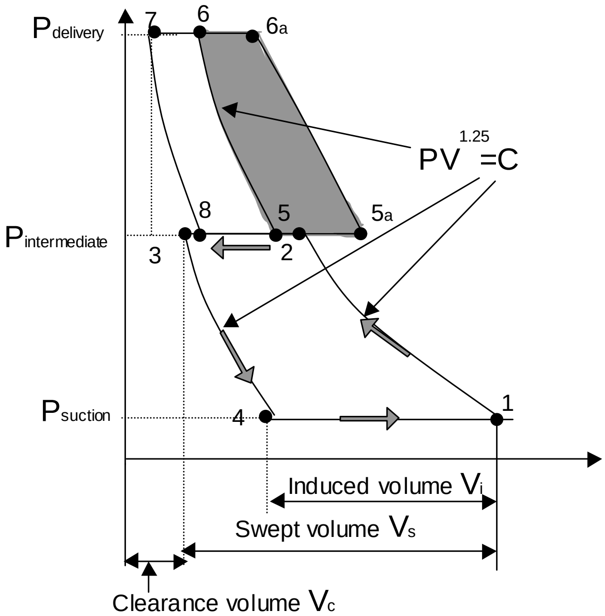
$$\text{Input power} = \frac{\text{power to compress air}}{\text{mechanical efficiency}} = \frac{8.973}{0.8} = 11.21 \ \text{kW}$$

Power required to drive the compressor is 11.21 kW

Worked example three
A two stage single acting air compressor has a clearance volume of 1 % of the swept volume and delivers 4.895 m^3 of air per minute at NTP.
Inlet conditions are 0.95 bar 25°C.
The delivery pressure is 14 bar with a second stage inlet pressure of 3.8 bar.
The index of expansion and compression in both stages is 1.25.
For air R= 287 J/kgK and c_p =1005 J/kgK

Calculate

a) The power saved by intercooling the air to 25°C
b) The heat removed in the inter-cooler



The power saved by inter-cooling is that shown shaded on the second stage portion of the diagram and is the difference between the second stage powers calculated at the different inlet temperatures.
All we have to do then is calculate the second stage powers, however I will calculate the first stage power also to show that it is not the same as the second stage since the pressure ratios are not the same.
The fact that the stage power or work is the same only when the inlet temperatures and the pressure ratios are the same must be clear.

Since we are dealing with a volume of air delivered at normal temperatures and pressures it is easier to convert this to a mass, which will then remain constant through out the machine.

Using $pV = mRT$ $\quad m = \dfrac{pV}{RT}$ $\quad m = \dfrac{1.01325 \times 10^5 \times 4.895}{287 \times 288} = 6.0 \ \text{kg}/\text{minute}$

$$\text{mass flow} = \frac{6.0}{60} = 0.1 \ \text{kg}/\text{second}$$

To calculate the stage powers we use the power equation for each stage

$$\text{Stage Power} = \frac{n}{n-1}\left(\dot{m}RT_{inlet}\right)\left[\left(\frac{p_{outlet}}{p_{inlet}}\right)^{\frac{n-1}{n}} - 1\right]$$

$$\text{First Stage Power} = \frac{1.25}{1.25-1}\left(0.1 \times 287 \times 298\right)\left[\left(\frac{3.8}{0.95}\right)^{\frac{1.25-1}{1.25}} - 1\right]$$

$$\text{First Stage Power} = 13.663 \ \text{kW}$$

$$\text{Second Stage Power} = \frac{1.25}{1.25-1}\left(0.1 \times 287 \times 393\right)\left[\left(\frac{14}{3.8}\right)^{\frac{1.25-1}{1.25}} - 1\right]$$

$$\text{Second Stage Power} = 16.804 \ \text{kW}$$

The second stage power with perfect inter-cooling is given by

$$\text{Second Stage Power} = \frac{1.25}{1.25-1}\left(0.1 \times 287 \times 298\right)\left[\left(\frac{14}{3.8}\right)^{\frac{1.25-1}{1.25}} - 1\right]$$

$$\text{Second Stage Power} = 12.742 \ \text{kW}$$

The power saved is therefore the original 16.804 kW less the new 12.742 kW

Power saved = 4.061 kW